ΛΥΣΕΙΣ ΤΩΝ ΑΣΚΗΣΕΩΝ ΤΗΣ ΤΡΙΓΩΝΟΜΕΤΡΙΑΣ

Ε'. ΓΥΜΝΑΣΙΟΥ

ΤΟΥ ΕΓΚΕΚΡΙΜΕΝΟΥ ΒΙΒΛΙΟΥ ΤΟΥ Ο.Ε.Δ.Β.

ΥΠΟ

ΙΩΑΝΝΟΥ Φ. ΠΑΝΑΚΗ

ΕΚΔΟΣΕΙΣ Ι. ΣΙΔΕΡΗΣ ΑΘΗΝΑΙ

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ΜΑΘΗΜΑΤΙΚΟΥ
ΤΗΣ ΙΩΝΙΔΕΙΟΥ ΠΡΟΤΥΠΟΥ ΣΧΟΛΗΣ ΠΕΙΡΑΙΩΣ

ΑΣΚΗΣΕΙΣ ΤΡΙΓΩΝΟΜΕΤΡΙΑΣ

**Περιέχει
τὰς λύσεις τῶν ἀσκήσεων τῆς Τριγωνομετρίας
τοῦ ἰδίου τῆς Ε′ τάξεως τοῦ Γυμνασίου, Θετικῆς κατευθύνσεως**

ΕΚΔΟΤΙΚΟΣ ΟΙΚΟΣ
"Ι. ΣΙΔΕΡΗΣ"
ΑΘΗΝΑΙ

Πᾶν γνήσιον ἀντίτυπον δέον ἀπαραιτήτως νὰ φέρῃ τὴν ἰδιόχειρον ὑπογραφὴν τοῦ Συγγραφέως.

ΑΣΚΗΣΕΙΣ ΤΡΙΓΩΝΟΜΕΤΡΙΑΣ

ΚΕΦΑΛΑΙΟΝ Ι.

1. Νὰ ὑπολογισθοῦν οἱ τριγωνομετρικοὶ ἀριθμοὶ τῆς γωνίας 105°.

Λύσις : Ἔχομεν διαδοχικῶς :

$$\eta\mu\ 105^\circ = \eta\mu\ (60^\circ + 45^\circ) = \eta\mu\ 60^\circ\ \sigma\upsilon\nu\ 45^\circ + \eta\mu\ 45^\circ\ \sigma\upsilon\nu\ 60^\circ =$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

$$\sigma\upsilon\nu\ 105^\circ = \sigma\upsilon\nu\ (60^\circ + 45^\circ) = \sigma\upsilon\nu 60^\circ \sigma\upsilon\nu 45^\circ - \eta\mu\ 45^\circ \cdot \eta\mu 60^\circ =$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4} = -\frac{\sqrt{6} - \sqrt{2}}{4}.$$

Κατ' ἀκολουθίαν : $\varepsilon\varphi\ 105^\circ = -\dfrac{\sqrt{6} + \sqrt{2}}{\sqrt{6} - \sqrt{2}} = -\dfrac{6 + 2 + 4\sqrt{3}}{4} = -(2 + \sqrt{3})$

καὶ $\sigma\varphi\ 105^\circ = -\dfrac{1}{2 + \sqrt{3}} = -\dfrac{2 - \sqrt{3}}{1} = -(2 - \sqrt{3}).$

2. Ἐὰν $0 < \alpha < \frac{\pi}{2}$, $0 < \beta < \frac{\pi}{2}$ καὶ ημ $\alpha = \frac{3}{5}$, συν $\beta = \frac{9}{41}$, νὰ ὑπολογισθοῦν αἱ παραστάσεις :

$$\eta\mu(\alpha - \beta), \quad \sigma\upsilon\nu(\alpha + \beta), \quad \varepsilon\varphi(\alpha - \beta), \quad \sigma\varphi(\alpha + \beta).$$

Λύσις : Θὰ εἶναι :

$$\sigma\upsilon\nu\ \alpha = +\sqrt{1 - \eta\mu^2\alpha} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

καὶ

$$\eta\mu\ \beta = +\sqrt{1 - \sigma\upsilon\nu^2\beta} = \sqrt{1 - \frac{81}{1681}} = \frac{40}{41}.$$

Κατ' ἀκολουθίαν :

$\varepsilon\varphi\alpha = \dfrac{3}{4}$ καὶ $\varepsilon\varphi\beta = \dfrac{40}{9}$, ὅτε : $\sigma\varphi\alpha = \dfrac{4}{3}$ καὶ $\sigma\varphi\beta = \dfrac{9}{40}$. Ἄρα :

$$\eta\mu(\alpha - \beta) = \eta\mu\alpha\ \sigma\upsilon\nu\beta - \eta\mu\beta\ \sigma\upsilon\nu\alpha = \frac{3}{5} \cdot \frac{9}{41} - \frac{40}{41} \cdot \frac{4}{5} = \frac{27 - 160}{205} = -\frac{133}{205}.$$

$$\sigma\upsilon\nu(\alpha+\beta)=\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta-\eta\mu\alpha\eta\mu\beta=\frac{4}{5}\cdot\frac{9}{41}-\frac{3}{5}\cdot\frac{40}{41}=\frac{36-120}{205}=-\frac{84}{205}.$$

$$\varepsilon\varphi(\alpha-\beta)=\frac{\varepsilon\varphi\alpha-\varepsilon\varphi\beta}{1+\varepsilon\varphi\alpha\varepsilon\varphi\beta}=\frac{\frac{3}{4}-\frac{40}{9}}{1+\frac{3}{4}\cdot\frac{40}{9}}=-\frac{31}{39}.$$

$$\sigma\varphi(\alpha+\beta)=\frac{\sigma\varphi\alpha\sigma\varphi\beta-1}{\sigma\varphi\alpha+\sigma\varphi\beta}=\frac{\frac{4}{3}\cdot\frac{9}{40}-1}{\frac{4}{3}+\frac{9}{40}}=\frac{36-120}{160+27}=-\frac{84}{187}.$$

3. Ἐὰν $\frac{\pi}{2}<\alpha<\pi$, $\frac{3\pi}{2}<\beta<2\pi$ καὶ $\eta\mu\alpha=\frac{15}{17}$, $\sigma\upsilon\nu\beta=\frac{12}{13}$, νὰ ὑπολογισθοῦν αἱ παραστάσεις:

$$\eta\mu(\alpha+\beta),\quad \sigma\upsilon\nu(\alpha-\beta),\quad \varepsilon\varphi(\alpha+\beta),\quad \sigma\varphi(\alpha-\beta).$$

Λύσις. Θὰ εἶναι:

$$\sigma\upsilon\nu\alpha=-\sqrt{1-\eta\mu^2\alpha}=-\sqrt{1-\frac{225}{289}}=-\frac{8}{17}\ \text{καὶ}\ \varepsilon\varphi\alpha=-\frac{\frac{15}{17}}{\frac{8}{17}}=-\frac{15}{8}$$

$$\eta\mu\beta=-\sqrt{1-\sigma\upsilon\nu^2\beta}=-\sqrt{1-\frac{144}{169}}=-\frac{5}{13}\ \text{καὶ}\ \varepsilon\varphi\beta=-\frac{\frac{5}{13}}{\frac{12}{13}}=-\frac{5}{12}$$

ὁπότε: $\sigma\varphi\alpha=-\frac{8}{15}$ καὶ $\sigma\varphi\beta=-\frac{12}{5}$. Ἄρα:

$$\eta\mu(\alpha+\beta)=\eta\mu\alpha\sigma\upsilon\nu\beta+\eta\mu\beta\sigma\upsilon\nu\alpha=\frac{15}{17}\cdot\frac{12}{13}+\left(-\frac{5}{13}\right)\cdot\left(-\frac{15}{17}\right)=$$

$$=\frac{180}{221}+\frac{75}{221}=\frac{255}{221}.$$

$$\sigma\upsilon\nu(\alpha-\beta)=\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu\alpha\eta\mu\beta=-\frac{8}{17}\cdot\frac{12}{13}+\frac{15}{17}\cdot\left(-\frac{5}{13}\right)=$$

$$=-\frac{96}{221}-\frac{75}{221}=-\frac{171}{221}.$$

$$\varepsilon\varphi(\alpha+\beta)=\frac{\varepsilon\varphi\alpha+\varepsilon\varphi\beta}{1-\varepsilon\varphi\alpha\cdot\varepsilon\varphi\beta}=\frac{-\frac{15}{8}-\frac{5}{12}}{1-\frac{15}{8}\cdot\frac{5}{12}}=\frac{-220}{11}=-\frac{220}{11}$$

$$\sigma\varphi(\alpha-\beta)=\frac{\sigma\varphi\alpha\sigma\varphi\beta+1}{\sigma\varphi\beta-\sigma\varphi\alpha}=\frac{-\frac{8}{15}\cdot\left(\frac{12}{5}\right)+1}{-\frac{12}{5}+\frac{8}{15}}=\frac{96+75}{-180+40}=-\frac{171}{140}.$$

4. Ἐὰν $0<\alpha<\frac{\pi}{2}$, $\frac{\pi}{2}<\beta<\pi$ καὶ $συνα=\frac{1}{\sqrt{2}}$, $συνβ=-\frac{3}{5}$, νὰ ὑπολογισθοῦν αἱ παραστάσεις :

1) ημ(α+β), συν(α−β), εφ(α−β), σφ(α+β).

2) Ἐὰν $εφx=\frac{β}{α}$, νὰ ἀποδειχθῇ ὅτι : ασυν2x+βημ2x=α.

Λύσις. 1. Εἶναι :

$$ημα=+\sqrt{1-συν^2α}=\sqrt{1-\frac{1}{2}}=\frac{1}{\sqrt{2}}, \quad ὅτε \quad εφα=1 \text{ καὶ } σφα=1.$$

$$ημβ=+\sqrt{1-συν^2β}=\sqrt{1-\frac{9}{25}}=\frac{4}{5}, \text{ ὅτε } εφβ=-\frac{3}{4} \text{ καὶ } σφβ=-\frac{4}{3}.$$

Κατ' ἀκολουθίαν :

$$ημ(α+β)=ημασυνβ+ημβσυνα=\frac{\sqrt{2}}{2}\left(-\frac{3}{5}\right)+\frac{4}{5}\cdot\frac{\sqrt{2}}{2}=-\frac{3\sqrt{2}}{10}+\frac{4\sqrt{2}}{10}=\frac{\sqrt{2}}{10}$$

$$συν(α-β)=συνασυνβ+ημαημβ=\frac{\sqrt{2}}{2}\cdot\left(-\frac{3}{5}\right)+\frac{\sqrt{2}}{2}\cdot\frac{4}{5}=-\frac{3\sqrt{2}}{10}+\frac{4\sqrt{2}}{10}=\frac{\sqrt{2}}{10}$$

$$εφ(α-β)=\frac{εφα-εφβ}{1+εφαεφβ}=\frac{1+\frac{4}{3}}{1+1\left(-\frac{3}{4}\right)}=\frac{7}{1}=7.$$

$$σφ(α+β)=\frac{σφασφβ-1}{σφα+σφβ}=\frac{1\cdot\left(-\frac{4}{3}\right)-1}{1-\frac{4}{3}}=\frac{-7}{-1}=-7.$$

2. Ἔχομεν διαδοχικῶς :

$$ασυν2x+βημ2x=α\left(συν2x+\frac{β}{α}ημ2x\right)=α(συν2x+εφxημ2x)=$$

$$=α\left(συν2x+\frac{ημx}{συνx}ημ2x\right)=\frac{α}{συνx}(συν2xσυνx+ημxημ2x)=$$

$$=\frac{α}{συνx}[συν(2x-x)]=\frac{α}{συνx}συνx=α.$$

5. Ἐὰν $\pi<\alpha<\frac{3\pi}{2}$, $0<\beta<\frac{\pi}{2}$ καὶ $εφα=\frac{8}{15}$, $συνβ=\frac{4}{5}$, νὰ ὑπολογισθοῦν αἱ παραστάσεις :

ημ(α−β), συν(α+β), εφ(α+β), σφ(α−β).

Λύσις. Εἶναι :

$$ημα=\frac{εφα}{-\sqrt{1+εφ^2α}}=\frac{\frac{8}{15}}{-\sqrt{1+\frac{64}{225}}}=-\frac{8}{17},$$

καὶ ἄρα

$$συνα=-\frac{15}{17} \text{ καὶ } σφα=\frac{15}{8}.$$

Ἐπίσης: $ημβ = \sqrt{1-συν^2β} = \sqrt{1-\frac{16}{25}} = \frac{3}{5}$,

ὁπότε: $εφβ = \frac{3}{4}$ καὶ $σφβ = \frac{4}{3}$.

Ἄρα:

$$ημ(α-β) = ημασυνβ - ημβσυνα = -\frac{8}{17}\cdot\frac{4}{5} - \frac{3}{5}\cdot\left(-\frac{15}{17}\right) = \frac{13}{85}$$

$$συν(α+β) = συνασυνβ - ημαημβ = -\frac{15}{17}\cdot\frac{4}{5} + \frac{8}{17}\cdot\frac{3}{5} = -\frac{36}{85}$$

$$εφ(α+β) = \frac{εφα+εφβ}{1-εφαεφβ} = \frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\cdot\frac{3}{4}} = \frac{77}{36}$$

$$σφ(α-β) = \frac{σφασφβ+1}{σφβ-σφα} = \frac{\frac{15}{8}\cdot\frac{4}{3}+1}{\frac{4}{3}-\frac{15}{8}} = \frac{84}{-13} = -\frac{84}{13}.$$

6. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ταυτότητες:

1. **$ημ(α-β)συνβ + ημβσυν(α-β) \equiv ημα$.**

Λύσις. Θὰ ἔχωμεν διαδοχικῶς:

$$ημ(α-β)συνβ + ημβσυν(α-β) \equiv ημ[(α-β)+β] \equiv ημ(α-β+β) \equiv ημα.$$

2. **$συν(α-β)συν(α+β) - ημ(α-β)ημ(α+β) \equiv συν2α$.**

Λύσις. Ἐὰν θέσωμεν $α-β=A$ καὶ $α+β=B$ τὸ α′ μέλος γίνεται:

$$συν(α-β)συν(α+β) - ημ(α-β)ημ(α+β) \equiv συνAσυνB - ημAημB \equiv$$
$$\equiv συν(A+B) = συν[(α-β)+(α+β)] \equiv συν2α.$$

3. **$ημ(60^\circ-α)συν(30^\circ+α) + ημ(30^\circ+α)συν(60^\circ-α) \equiv 1$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ(60^\circ-α)συν(30^\circ+α) + ημ(30^\circ+α)συν(60^\circ-α) \equiv ημ[(60^\circ-α)+(30^\circ+α)] \equiv$$
$$\equiv ημ90^\circ \equiv 1.$$

4. **$συν(45^\circ-α)συν(45^\circ-β) - ημ(45^\circ-α)ημ(45^\circ-β) \equiv ημ(α+β)$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν(45^\circ-α)συν(45^\circ-β) - ημ(45^\circ-α)ημ(45^\circ-β) \equiv$$
$$= συν[(45^\circ-α)+(45^\circ-β)] \equiv συν[90^\circ-(α+β)] \equiv ημ(α+β).$$

5. **$ημ(45^\circ+α)συν(45^\circ-β) + συν(45^\circ+α)ημ(45^\circ-β) \equiv συν(α-β)$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ(45^\circ+α)συν(45^\circ-β) + συν(45^\circ+α)ημ(45^\circ-β) \equiv$$
$$\equiv ημ[(45^\circ+α)+(45^\circ-β)] \equiv ημ[90^\circ+(α-β)] \equiv συν(α-β).$$

6. **$συν(36^\circ-α)συν(36^\circ+α) + συν(54^\circ+α)συν(54^\circ-α) \equiv συν2α$.**

Λύσις. Ἐπειδὴ $συν(36^0-α)\equiv ημ(54^0+α)$, $συν(36^0+α)\equiv ημ(54^0-α)$ θὰ ἔχωμεν:

$$συν(36^0-α)συν(36^0+α)+συν(54^0+α)συν(54^0-α)\equiv$$
$$\equiv ημ(54^0+α)ημ(54^0-α)+συν(54^0+α)συν(54^0-α)\equiv$$
$$\equiv συν[(54^0+α)-(54^0-α)]\equiv συν2α.$$

7. $\mathbf{συν(30^0+α)συν(30^0-α)-ημ(30^0+α)ημ(30^0-α)\equiv \frac{1}{2}}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν(30^0+α)συν(30^0-α)-ημ(30^0+α)ημ(30^0-α)\equiv$$
$$\equiv συν[(30^0+α)+(30^0-α)]\equiv συν60^0\equiv \frac{1}{2}.$$

8. $\mathbf{ημ(ν+1)Αημ(ν-1)Α+συν(ν+1)Ασυν(ν-1)Α\equiv συν2Α}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ(ν+1)Αημ(ν-1)Α+συν)ν+1)Α(συν(ν-1)Α\equiv συν[(ν+1)Α-(ν-1)Α]\equiv$$
$$\equiv συν[νΑ+Α-νΑ+Α]\equiv συν2Α.$$

9. $\mathbf{ημ(ν+1)Αημ(ν+2)Α+συν(ν+1)Ασυν(ν+2)Α\equiv συνΑ}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ(ν+1)Αημ(ν+2)Α+συν(ν+1)Ασυν(ν+2)Α\equiv συν[(ν+1)Α-(ν+2)Α]\equiv$$
$$\equiv συν[νΑ+Α-νΑ-2Α[\equiv συν(-Α)\equiv συνΑ.$$

10. $\mathbf{εφ(β-γ)+εφ(γ-α)+εφ(α-β)=εφ(β-γ)εφ(γ-α)εφ(α-β)}$.

Λύσις. Ἐὰν τεθῇ $β-γ=x$, $γ-α=y$, $α-β=ω$, θὰ εἶναι:

$x+y+ω=β-γ+γ-α+α-β=0$ ἢ $x+y=-ω$, ὅτε:

$$εφ(x+y)=εφ(-ω)=-εφω \quad ἢ \quad \frac{εφx+εφy}{1-εφxεφy}=-εφω$$

ἐξ οὗ: $$εφx+εφy+εφω=εφxεφyεφω$$

ἢ $$εφ(β-γ)+εφ(γ-α)+εφ(α-β)=εφ(β-γ)εφ(γ-α)εφ(α-β).$$

7. Νὰ ἀποδειχθῇ ὅτι:

$$\mathbf{συν(α+β)συν(α-β)\equiv συν^2α-ημ^2β\equiv συν^2β-ημ^2α.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν(α+β)συν(α-β)\equiv(συνασυνβ-ημαημβ)(συνασυνβ+ημαημβ)$$
$$\equiv συν^2ασυν^2β-ημ^2αημ^2β)$$
$$\equiv συν^2α(1-ημ^2β)-(1-συν^2α)ημ^2β$$
$$\equiv συν^2α-συν^2αημ^2β-ημ^2β+συν^2αημ^2β$$
$$\equiv συν^2α-ημ^2β\equiv(1-ημ^2α)-(1-συν^2β)\equiv ημ^2α.$$

8. Νὰ ἀποδειχθῇ ὅτι:

1. $$\mathbf{\frac{ημ(α-β)}{συνασυνβ}+\frac{ημ(β-γ)}{συνβσυνγ}+\frac{ημ(γ-α)}{συνγσυνα}=0.}$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α΄ μέλος, πρέπει: $συνασυνβσυνγ\neq 0$:

$$\left.\begin{array}{ll} ἢ & συνα\neq 0 \\ καὶ & συνβ\neq 0 \\ καὶ & συνγ\neq 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} α\neq kπ+\frac{π}{2} \\ β\neq k_1π+\frac{π}{2} \\ γ\neq k_2π+\frac{π}{2} \end{array}\right\},\quad (k, k_1, k_2)\in \mathbf{Z}$$

Καλοῦμεν Α τὸ πρῶτον μέλος καὶ ἔχομεν:

$$\frac{ημ(α-β)}{συνασυνβ} \equiv \frac{ημασυνβ - ημβσυνα}{συνασυνβ} \equiv \frac{ημασυνβ}{συνασυνβ} - \frac{ημβσυνα}{συνασυνβ} \equiv εφα - εφβ$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς τῶν α,β,γ, λαμβάνομεν:

$$Α \equiv εφα - εφβ + εφβ - εφγ + εφγ - εφα = 0.$$

2. $$\frac{ημ(β-γ)}{ημβημγ} + \frac{ημ(γ-α)}{ημγημα} + \frac{ημ(α-β)}{ημαημβ} = 0.$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α΄ μέλος, πρέπει: $ημαημβημγ \neq 0$

ἢ $ημα \neq 0$
καὶ $ημβ \neq 0$
καὶ $ημγ \neq 0$

$$\left.\begin{array}{l} ημα \neq 0 \\ ημβ \neq 0 \\ ημγ \neq 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} α \neq kπ \\ β \neq k_1π \\ γ \neq k_2π \end{array}\right\}, \quad (k, k_1, k_2) \in \mathbf{Z})$$

Καλοῦντες Α τὸ α΄ μέλος καὶ παρατηροῦντες ὅτι:

$$\frac{ημ(β-γ)}{ημβημγ} = \frac{ημβσυνγ - ημγσυνβ}{ημβημγ} = \frac{ημβσυνγ}{ημβημγ} - \frac{ημγσυνβ}{ημβημγ} = σφγ - σφβ$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς, θὰ ἔχωμεν:

$$Α = σφγ - σφβ + σφα - σφγ + σφβ - σφα = 0.$$

9. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ταυτότητες:

1. $$\frac{ημ(α+β)ημ(α-β)}{συν^2ασυν^2β} = εφ^2α - εφ^2β.$$

Λύσις. Ἔχοντες ὑπ᾽ ὄψει τὸν τύπον (11), λαμβάνομεν:

$$\frac{ημ(α+β)ημ(α-β)}{συν^2ασυν^2β} = \frac{ημ^2α - ημ^2β}{συν^2ασυν^2β} = \frac{ημ^2ασυν^2β - ημ^2βσυν^2α}{συν^2ασυν^2β} =$$

$$= \frac{ημ^2ασυν^2β}{συν^2ασυν^2β} - \frac{ημ^2βσυν^2α}{συν^2ασυν^2β} = εφ^2α - εφ^2β,$$

ἂν $α \neq kπ + \frac{π}{2}$ καὶ $β \neq k_1π + \frac{π}{2}$, $(k, k_1) \in \mathbf{Z}$.

2. $$\frac{εφα + εφβ}{εφα - εφβ} = \frac{ημ(α+β)}{ημ(α-β)}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α - β \neq kπ$, $k \in \mathbf{Z}$

$$\frac{εφα + εφβ}{εφα - εφβ} = \frac{\frac{ημα}{συνα} + \frac{ημβ}{συνβ}}{\frac{ημα}{συνα} - \frac{ημβ}{συνβ}} = \frac{ημασυνβ + ημβσυνα}{ημασυνβ - ημβσυνα} = \frac{ημ(α+β)}{ημ(α-β)}.$$

3. $$\frac{2ημ(α+β)}{συν(α+β) + συν(α-β)} = εφα + εφβ.$$

Λύσις. Εἶναι:

$$\frac{2ημ(α+β)}{συν(α+β) + συν(α-β)} = \frac{2(ημασυνβ + ημβσυνα)}{συνασυνβ - ημαημβ + συνασυνβ + ημαημβ}$$

$$= \frac{2(ημασυνβ + ημβσυνα}{2συνασυνβ} = \frac{ημασυνβ}{συνασυνβ} + \frac{ημβσυνα}{συνασυνβ} = εφα + εφβ,$$

ἂν $\alpha \neq k\pi + \frac{\pi}{2}, \quad \beta \neq k_1\pi + \frac{\pi}{2}, \quad (k, k_1 \in \mathbf{Z})$

4. $$\frac{\varepsilon\varphi^2 2\alpha - \varepsilon\varphi^2\alpha}{1 - \varepsilon\varphi^2 2\alpha\varepsilon\varphi^2\alpha} = \varepsilon\varphi 3\alpha\varepsilon\varphi\alpha.$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α΄ μέλος, πρέπει:
$\alpha \neq k\frac{\pi}{2} + \frac{\pi}{4}$ καὶ $\alpha \neq k_1 k + \frac{\pi}{2}$ $(k, k_1 \in \mathbf{Z})$. Ἔχομεν δὲ διαδοχικῶς:

$$\frac{\varepsilon\varphi^2 2\alpha - \varepsilon\varphi^2\alpha}{1 - \varepsilon\varphi^2 2\alpha\varepsilon\varphi^2\alpha} \equiv \frac{\varepsilon\varphi 2\alpha + \varepsilon\varphi\alpha}{1 - \varepsilon\varphi 2\alpha\varepsilon\varphi\alpha} \cdot \frac{\varepsilon\varphi 2\alpha - \varepsilon\varphi\alpha}{1 + \varepsilon\varphi 2\alpha\varepsilon\varphi\alpha} = \varepsilon\varphi(2\alpha + \alpha)\varepsilon\varphi(2\alpha - \alpha) = \varepsilon\varphi 3\alpha\varepsilon\varphi\alpha.$$

5. $$\frac{\sigma\varphi 4\alpha\sigma\varphi 3\alpha + 1}{\sigma\varphi 3\alpha - \sigma\varphi 4\alpha} = \sigma\varphi\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq k\pi$, $k \in \mathbf{Z}$

$$\frac{\sigma\varphi 4\alpha\sigma\varphi 3\alpha + 1}{\sigma\varphi 3\alpha - \sigma\varphi 4\alpha} = \sigma\varphi(4\alpha - 3\alpha) = \sigma\varphi\alpha.$$

6. $$(\sigma\upsilon\nu\alpha - \eta\mu\alpha)(\sigma\upsilon\nu 2\alpha - \eta\mu 2\alpha) \equiv \sigma\upsilon\nu\alpha - \eta\mu 3\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$(\sigma\upsilon\nu\alpha - \eta\mu\alpha)(\sigma\upsilon\nu 2\alpha - \eta\mu 2\alpha) \equiv (\sigma\upsilon\nu\alpha\sigma\upsilon\nu 2\alpha - \eta\mu\alpha\sigma\upsilon\nu 2\alpha) - (\sigma\upsilon\nu\alpha\eta\mu 2\alpha - \eta\mu\alpha\eta\mu 2\alpha) \equiv$$
$$\equiv (\sigma\upsilon\nu\alpha\sigma\upsilon\nu 2\alpha + \eta\mu\alpha\eta\mu 2\alpha) - (\eta\mu\alpha\sigma\upsilon\nu 2\alpha + \sigma\upsilon\nu\alpha\eta\mu 2\alpha) \equiv$$
$$\equiv \sigma\upsilon\nu(2\alpha - \alpha) - \eta\mu(2\alpha + \alpha) \equiv \sigma\upsilon\nu\alpha - \eta\mu 3\alpha.$$

7. $$\frac{\varepsilon\varphi(\alpha - \beta) + \varepsilon\varphi\beta}{1 - \varepsilon\varphi(\alpha - \beta)\varepsilon\varphi\beta} = \varepsilon\varphi\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha - \beta \neq k\pi + \frac{\pi}{2}$ καὶ $\beta \neq k_1\pi + \frac{\pi}{2}$,

καθὼς καὶ $\alpha \neq k_2\pi + \frac{\pi}{2}$, ἔνθα $(k, k_1, k_2) \in \mathbf{Z}$.

$$\frac{\varepsilon\varphi(\alpha - \beta) + \varepsilon\varphi\beta}{1 - \varepsilon\varphi(\alpha - \beta)\varepsilon\varphi\beta} = \varepsilon\varphi[(\alpha - \beta) - \beta] = \varepsilon\varphi\alpha.$$

10. Νὰ ἀποδειχθῇ ὅτι:

$$\sigma\upsilon\nu^2 x + \sigma\upsilon\nu^2(120^\circ + x) + \sigma\upsilon\nu^2(120^\circ - x) \equiv \frac{3}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu^2 x + \sigma\upsilon\nu^2(120^\circ + x) + \sigma\upsilon\nu^2(120^\circ - x) \equiv \sigma\upsilon\nu^2 x + (\sigma\upsilon\nu 120^\circ\sigma\upsilon\nu x - \eta\mu x\eta\mu 120^\circ)^2 +$$
$$+ (\sigma\upsilon\nu 120^\circ\sigma\upsilon\nu x + \eta\mu x\eta\mu 120^\circ)^2 \equiv$$
$$\equiv \sigma\upsilon\nu^2 x + \left(-\frac{1}{2}\sigma\upsilon\nu x - \frac{\sqrt{3}}{2}\eta\mu x\right)^2 + \left(-\frac{1}{2}\sigma\upsilon\nu x + \frac{\sqrt{3}}{2}\eta\mu x\right)^2$$
$$\equiv \sigma\upsilon\nu^2 x + \frac{1}{4}\sigma\upsilon\nu^2 x + \frac{3}{4}\eta\mu^2 x + \frac{\sqrt{3}}{2}\eta\mu x\sigma\upsilon\nu x +$$
$$+ \frac{1}{4}\sigma\upsilon\nu^2 x + \frac{3}{4}\eta\mu^2 x + \frac{\sqrt{3}}{2}\eta\mu x\sigma\upsilon\nu x \equiv \frac{3}{2}\sigma\upsilon\nu^2 x + \frac{3}{2}\eta\mu^2 x \equiv$$
$$\equiv \frac{3}{2}(\sigma\upsilon\nu^2 x + \eta\mu^2 x) \equiv \frac{3}{2} \cdot 1 \equiv \frac{3}{2}.$$

11. Νὰ ἀποδειχθῇ ὅτι αἱ παραστάσεις :

1. $A \equiv συν^2x - 2συνασυνxσυν(α+x) + συν^2(α+x)$,

2. $B \equiv συν^2x - 2ημασυνxημ(α+x) + ημ^2(α+x)$,

εἶναι ἀνεξάρτητοι τοῦ x. Ποῖον εἶναι τὸ ἄθροισμα τῶν παραστάσεων τούτων ;

Λύσις. Ἔχομεν διαδοχικῶς :

$$A \equiv συν^2x - συν(α+x)[2συνασυνx - συν(α+x)]$$
$$\equiv συν^2x - συν(α+x)(2συνασυνx - συνασυνx + ημαημx)$$
$$\equiv συν^2x - συν(α+x)(συνασυνx + ημαημx)$$
$$\equiv συν^2x - συν(α+x)συν)(α-x) \equiv συν^2x - (συν^2x - ημ^2α) \equiv ημ^2α.$$

2. $$B \equiv συν^2x - 2ημασυνxημ(α+x) + ημ^2(α+x)$$
$$\equiv συν^2x - ημ(α+x)[2ημασυνx - ημ(α+x)]$$
$$\equiv συν^2x - ημ(α+x)(2ημασυνx - ημασυνx - ημxσυνα)$$
$$\equiv συν^2x - ημ(α+x)(ημασυνx - ημxσυνα)$$
$$\equiv συν^2x - ημ(α+x)ημ(α-x) \equiv συν^2x - (ημ^2α - ημ^2x) \equiv$$
$$\equiv συν^2x + ημ^2x - ημ^2α \equiv 1 - ημ^2α \equiv συν^2α.$$

Κατ' ἀκολουθίαν : $A+B \equiv ημ^2α + συν^2α \equiv 1$.

12. Ἐὰν $α+β=45^0$, νὰ ἀποδειχθῇ ὅτι :

1. $$(1+εφα)(1+εφβ)=2$$

2. **Ἐὰν $ημx - ημy = α$, $συνx + συνy = β$, νὰ ὑπολογισθῇ τὸ $συν(x+y)$ καὶ νὰ γίνῃ διερεύνησις.**

Λύσις. 1. Ἐκ τῆς σχέσεως $α+β=45^0$ ἔχομεν :

$$εφ(α+β) = εφ45^0 = 1 \quad \text{ἢ} \quad \frac{εφα+εφβ}{1-εφαεφβ} = 1 \quad \text{ἢ} \quad εφα+εφβ = 1-εφαεφβ \quad \text{ἢ}$$

$$1+εφα+εφβ+εφαεφβ = 2 \quad \text{ἢ} \quad (1+εφα)(1+εφβ) = 2.$$

2. Ἐκ τῶν $ημx - ημy = α$ καὶ $συνx + συνy = β$, λαμβάνομεν :

$$\left.\begin{array}{r} ημ^2x + ημ^2y - 2ημxημy = α^2 \\ συν^2x + συν^2y + 2συνxσυνy = β^2 \end{array}\right\} \Longrightarrow 2 + 2(συνxσυνy - ημxημy) = α^2+β^2$$

$$\text{ἢ} \qquad 2+2συν(x+y) = α^2+β^2 \quad \text{ἢ} \quad 2συν(x+y) = α^2+β^2-2 \qquad (1)$$

Διερεύνησις : Διὰ νὰ ὑφίσταται ἡ (1), πρέπει :

$$\left|\frac{α^2+β^2-2}{2}\right| \leqq 1 \quad \text{ἢ} \quad |α^2+β^2-2| \leqslant 2 \quad \text{ἢ} \quad (α^2+β^2-2)^2 \leqslant 4$$

$$\text{ἢ} \qquad (α^2+β^2)(α^2+β^2-4) \leqslant 0 \Longrightarrow α^2+β^2 \leqslant 4, \quad \text{ἂν} \ αβ \neq 0.$$

$$\left.\begin{array}{ll} \text{Ἄρα} & α^2 \leqslant 4-β^2 \Longrightarrow 4-β^2 \geqslant 0 \ \text{ἢ} \ β^2 \leqslant 4 \Longrightarrow -2 \leqslant β \leqslant 2 \\ \text{καὶ} & β^2 \leqslant 4-α^3 \Longrightarrow 4-α^2 \geqslant 0 \ \text{ἢ} \ α^2 \leqslant 4 \Longrightarrow -2 \leqslant α \leqslant 2 \end{array}\right\}.$$

13. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι Α+Γ=135, νὰ ἀποδειχθῇ ὅτι :

$$(1+σφA)(1+σφΓ) = 2.$$

Λύσις. Ἐκ τῆς σχέσεως $A+Γ=135^0$ λαμβάνομεν :

$$σφ(A+Γ) = σφ135^0 = -1 \quad \text{ἢ} \quad \frac{σφAσφΓ-1}{σφA+σφΓ} = -1$$

ἤ $\sigma\varphi A \sigma\varphi \Gamma - 1 = -\sigma\varphi A - \sigma\varphi\Gamma$ ἤ $\sigma\varphi A + \sigma\varphi\Gamma + \sigma\varphi A\sigma\varphi\Gamma = 1$

ἤ $1 + \sigma\varphi A + \sigma\varphi\Gamma + \sigma\varphi A\sigma\varphi\Gamma = 1$ ἤ $(1+\sigma\varphi A)(1+\sigma\varphi\Gamma) = 2.$

14. Ἐὰν $0 < \alpha < \frac{\pi}{2}$ καὶ $0 < \beta < \frac{\pi}{2}$ καὶ $\varepsilon\varphi\alpha = \frac{\sqrt{2}+1}{\sqrt{2}-1}$, $\varepsilon\varphi\beta = \frac{\sqrt{2}}{2}$ νὰ ἀποδειχθῇ ὅτι: $\alpha - \beta = 45^{\circ}$.

Λύσις. Ἔχομεν: $\varepsilon\varphi\alpha = \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{(\sqrt{2}+1)^2}{1} = 3+2\sqrt{2}$

$$\varepsilon\varphi(\alpha-\beta) = \frac{\varepsilon\varphi\alpha - \varepsilon\varphi\beta}{1+\varepsilon\varphi\alpha\varepsilon\varphi\beta} = \frac{3+2\sqrt{2} - \frac{\sqrt{2}}{2}}{1+(3+\sqrt{2})\frac{\sqrt{2}}{2}} = \frac{6+3\sqrt{2}}{6+3\sqrt{2}} = 1 = \varepsilon\varphi 45^{\circ}.$$

Ἄρα $\alpha - \beta = 45^{\circ}$.

15. Ἐὰν $\alpha+\beta+\gamma=\pi$, νὰ ἀποδειχθῇ ὅτι:

1. $$\sigma\varphi\frac{\alpha}{2} + \sigma\varphi\frac{\beta}{2} + \sigma\varphi\frac{\gamma}{2} = \sigma\varphi\frac{\alpha}{2}\sigma\varphi\frac{\beta}{2}\sigma\varphi\frac{\gamma}{2} \qquad (1)$$

Λύσις: Ἐκ τῆς σχέσεως $\alpha+\beta+\gamma=\pi \Longrightarrow \frac{\alpha}{2}+\frac{\beta}{2} = \frac{\pi}{2} - \frac{\gamma}{2}$ καὶ κατ᾿ ἀκολουθίαν:

$$\sigma\varphi\left(\frac{\alpha}{2}+\frac{\beta}{2}\right) = \sigma\varphi\left(\frac{\pi}{2}-\frac{\gamma}{2}\right) = \varepsilon\varphi\frac{\gamma}{2} = \frac{1}{\sigma\varphi\frac{\gamma}{2}}$$

ἤ $$\frac{\sigma\varphi\frac{\alpha}{2}\sigma\varphi\frac{\beta}{2} - 1}{\sigma\varphi\frac{\alpha}{2} + \sigma\varphi\frac{\beta}{2}} = \frac{1}{\sigma\varphi\frac{\gamma}{2}} \Longrightarrow \sigma\varphi\frac{\alpha}{2} + \sigma\varphi\frac{\beta}{2} + \sigma\varphi\frac{\gamma}{2} = \sigma\varphi\frac{\alpha}{2}\sigma\varphi\frac{\beta}{2}\sigma\varphi\frac{\gamma}{2}.$$

Ἀντιστρόφως: Πῶς συνδέονται τὰ τόξα, α, β, γ ἂν ἰσχύῃ ἡ (1);

Ἀπάντησις: Πρέπει: $\alpha+\beta+\gamma = (2\nu+1)\pi$, $\nu \in \mathbf{Z}$.

2. $$\sigma\varphi\alpha\sigma\varphi\beta + \sigma\varphi\beta\sigma\varphi\gamma + \sigma\varphi\gamma\sigma\varphi\alpha = 1 \qquad (2)$$

Λύσις. Ἐκ τῆς $\alpha+\beta+\gamma=\pi$, ἕπεται $\alpha+\beta=\pi-\gamma$ ἤ

$$\sigma\varphi(\alpha+\beta) = \sigma\varphi(\pi-\gamma) = -\sigma\varphi\gamma \quad \text{ἤ} \quad \frac{\sigma\varphi\alpha\sigma\varphi\beta - 1}{\sigma\varphi\alpha + \sigma\varphi\beta} = -\sigma\varphi\gamma$$

ἐξ οὗ: $$\sigma\varphi\alpha\sigma\varphi\beta + \sigma\varphi\beta\sigma\varphi\gamma + \sigma\varphi\gamma\sigma\varphi\alpha = 1.$$

Ἀντιστρόφως: Πῶς συνδέονται τὰ τόξα α, β, γ, ἄν ἰσχύῃ ἡ (2);

Πρέπει: $\boxed{\alpha+\beta+\gamma = k\pi}$, $k \in \mathbf{Z}$.

3. $$A = (\sigma\varphi\alpha+\sigma\varphi\beta)(\sigma\varphi\beta+\sigma\varphi\gamma)(\sigma\varphi\gamma+\sigma\varphi\alpha) = \sigma\tau\varepsilon\mu\alpha\sigma\tau\varepsilon\mu\beta\sigma\tau\varepsilon\mu\gamma.$$

Λύσις. Ἐπειδή: $\sigma\varphi\alpha + \sigma\varphi\beta = \frac{\sigma\upsilon\nu\alpha}{\eta\mu\alpha} + \frac{\sigma\upsilon\nu\beta}{\eta\mu\beta} = \frac{\eta\mu\alpha\sigma\upsilon\nu\beta + \eta\mu\beta\sigma\upsilon\nu\alpha}{\eta\mu\alpha\eta\mu\beta} =$

$$\frac{\eta\mu(\alpha+\beta)}{\eta\mu\alpha\eta\mu\beta} = \frac{\eta\mu(\pi-\gamma)}{\eta\mu\alpha\eta\mu\beta} = \frac{\eta\mu\gamma}{\eta\mu\alpha\eta\mu\beta}, \quad \text{ἄν} \quad \left.\begin{array}{l} \alpha \neq k\pi \\ \beta \neq k_1\pi \\ \gamma \neq k_2\pi \end{array}\right\} \quad \text{καὶ} \ (k, k_1, k_2) \in \mathbf{Z}$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς θὰ ἔχωμεν:

$$A \equiv \frac{ημγ}{ημαημβ} \cdot \frac{ημα}{ημβημγ} \cdot \frac{ημβ}{ημαημγ} = \frac{1}{ημα} \cdot \frac{1}{ημβ} \cdot \frac{1}{ημγ} = στεμαστεμβστεμγ.$$

4. $$\frac{\boldsymbol{συνα}}{\boldsymbol{ημβημγ}} + \frac{\boldsymbol{συνβ}}{\boldsymbol{ημγημα}} + \frac{\boldsymbol{συνγ}}{\boldsymbol{ημαημβ}} = \mathbf{2}.$$

Τὸ πρῶτον μέλος ἔχει ἔννοιαν, ὅταν $α \neq kπ$, $β \neq k_1π$, $γ \neq k_2π$, $(k, k_1, k_2) \in \mathbf{Z}$.

Λύσις, Εἶναι: $$\frac{συνα}{ημβημγ} = \frac{συν[π-(β+γ)]}{ημβημγ} = \frac{-συν(β+γ)}{ημβημγ} =$$

$$= \frac{ημβημγ - συνβσυνγ}{ημβημγ} = 1 - σφβσφγ$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς, θὰ ἔχωμεν:

$$B = 1 - σφβσφγ + 1 - σφγσφα + 1 - σφασφβ =$$
$$= 3 - (σφβσφγ + σφγσφα + σφασφβ) = 3 - 1 = 2.$$

5. $$\mathbf{Γ} \equiv \frac{\boldsymbol{σφα+σφβ}}{\boldsymbol{εφα+εφβ}} + \frac{\boldsymbol{σφβ+σφγ}}{\boldsymbol{εφβ+εφγ}} + \frac{\boldsymbol{σφγ+σφα}}{\boldsymbol{εφγ+εφα}} = \mathbf{1}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{σφα+σφβ}{εφα+εφβ} = \frac{\dfrac{συνα}{ημα} + \dfrac{συνβ}{ημβ}}{\dfrac{ημα}{συνα} + \dfrac{ημβ}{συνβ}} = \frac{\dfrac{ημασυνβ+ημβσυνα}{ημαημβ}}{\dfrac{ημασυνβ+ημβσυνα}{συνασυνβ}} = \frac{συνασυνβ}{ημαημβ} = σφασφβ.$$

Διὰ κυκλικῆς ἐναλλαγῆς θὰ ἔχωμεν (ἄσκ. 15,2).

$$Γ \equiv σφασφβ + σφβσφγ + σφγσφα = 1.$$

6. $$\boldsymbol{ημ^2α + ημ^2β + ημ^2γ - 2συνασυνβσυνγ = 2.}$$

Λύσις. Ἐκ τῆς δοθείσης σχέσεως $α+β+γ=π \Longrightarrow α+β=π-γ$ ἢ
$συν(α+β) = συν(π-γ) = -συνγ$ ἢ $συνασυνβ - ημαημβ = -συνγ$

ἢ $$συνασυνβ + συνγ = ημαημβ.$$

ἢ $$συν^2ασυν^2β + συν^2γ + 2συνασυνβσυνγ = ημ^2γημ^2β =$$
$$= (1-συν^2α)(1-συν^2β) = 1 - συν^2α - συν^2β + συν^2ασυν^2β$$

ἢ $$συν^2γ + 2συνασυνβσυνγ = 1 - συν^2α - συν^2β$$

ἢ $$1 - ημ^2γ + 2συνασυνβσυνγ = ημ^2α - 1 + ημ^2β$$

ἐξ οὗ: $$ημ^2α + ημ^2β + ημ^2γ - 2συνασυνβσυνγ = 2.$$

7. $$\boldsymbol{εφ2α + εφ2β + εφ2γ = εφ2αεφ2γεφ2γ.} \quad (1)$$

Λύσις. Ἐκ τῆς $α+β+γ=π \Longrightarrow 2α+2β = 2π-γ$ ἢ

$$εφ(2α+2β) = εφ(2π-2γ) = -εφ2γ$$ ἢ

$$\frac{εφ2α+εφ2β}{1-εφ2αεφ2β} = -εφ2γ \Longrightarrow εφ2α + εφ2β + εφ2γ = εφ2αεφ2βεφ2γ.$$

Ἡ (1) δὲν ἔχει ἔννοιαν διὰ $α = \frac{π}{4}$ ἢ $β = \frac{π}{4}$ ἢ $γ = \frac{π}{4}$.

8. $$\boldsymbol{συν^2 2α + συν^2 2β + συν^2 2γ = 1 + 2συν2ασυν2βσυν2γ.}$$

Λύσις. Ἔχομεν $\alpha+\beta+\gamma=\pi \Longrightarrow 2\alpha+2\beta=2\pi-2\gamma$ ἢ

$συν(2\alpha+2\beta)=συν(2\pi-2\gamma)=συν2\gamma$ ἢ $συν2\alpha συν2\beta-ημ2\alpha ημ2\beta=συν2\gamma$

ἢ $$συν2\alpha συν2\beta-συν2\gamma=ημ2\alpha ημ2\beta$$

ἢ $$συν^2 2\alpha συν^2 2\beta+συν^2 2\gamma-2συν2\alpha συν2\beta συν2\gamma=ημ^2 2\alpha ημ^2 2\beta=$$

$$=(1-συν^2 2\alpha)(1-συν^2 2\beta)=1-συν^2 2\alpha-συν^2 2\beta+συν^2 2\alpha συν^2 2\beta$$

ἐξ οὗ: $$συν^2 2\alpha+συν^2 2\beta+συν^2 2\gamma=1+2συν2\alpha συν2\beta συν2\gamma.$$

9. $\boldsymbol{ημ^2 2\alpha+ημ^2 2\beta+ημ^2 2\gamma+2συν2\alpha συν2\beta συν2\gamma=2.}$

Λύσις. Γνωρίζομεν ὅτι:

$$συν^2 2\alpha+συν^2 2\beta+συν^2 2\gamma=1+2συν2\alpha συν2\beta συν2\gamma.$$

ἢ $$1-ημ^2 2\alpha+1-ημ^2 2\beta+1-ημ^2 2\gamma=1+2συν2\alpha συν2\beta συν2\gamma$$

ἐξ οὗ: $$ημ^2 2\alpha+ημ^2 2\beta+ημ^2 2\gamma+2συν2\alpha συν2\beta συν2\gamma=2.$$

10. $\boldsymbol{ημ^2\frac{\alpha}{2}+ημ^2\frac{\beta}{2}+ημ^2\frac{\gamma}{2}=1-2ημ\frac{\alpha}{2}\,ημ\frac{\beta}{2}\,ημ\frac{\gamma}{2}.}$

Λύσις. Ἔχομεν $\alpha+\beta+\gamma=\pi \Longrightarrow \frac{\alpha}{2}+\frac{\beta}{2}=\frac{\pi}{2}-\frac{\gamma}{2}$

ἢ $$συν\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=συν\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)=ημ\frac{\gamma}{2}$$

ἢ $$συν\frac{\alpha}{2}συν\frac{\beta}{2}-ημ\frac{\alpha}{2}ημ\frac{\beta}{2}=ημ\frac{\gamma}{2}$$

ἢ $$συν\frac{\alpha}{2}συν\frac{\beta}{2}=ημ\frac{\alpha}{2}ημ\frac{\beta}{2}+ημ\frac{\gamma}{2}$$

ἢ $$συν^2\frac{\alpha}{2}συν^2\frac{\beta}{2}=ημ^2\frac{\alpha}{2}ημ^2\frac{\beta}{2}+ημ^2\frac{\gamma}{2}+2ημ\frac{\alpha}{2}ημ\frac{\beta}{2}ημ\frac{\gamma}{2}$$

ἢ $$\left(1-ημ^2\frac{\alpha}{2}\right)\left(1-ημ^2\frac{\beta}{2}\right)=ημ^2\frac{\alpha}{2}ημ^2\frac{\beta}{2}+ημ^2\frac{\gamma}{2}+2ημ\frac{\alpha}{2}ημ\frac{\beta}{2}ημ\frac{\gamma}{2}$$

ἐξ οὗ, μετὰ τὰς πράξεις:

$$ημ^2\frac{\alpha}{2}+ημ^2\frac{\beta}{2}+ημ^2\frac{\gamma}{2}=1-2ημ\frac{\alpha}{2}ημ\frac{\beta}{2}συν\frac{\alpha}{2}.$$

16. Νὰ ἀποδειχθῇ ὅτι:

1. $\boldsymbol{συν^2\alpha+συν^2(60^0+\alpha)+συν^2(60^0-\alpha)\equiv\frac{3}{2}}$

2. $\boldsymbol{ημ^2\alpha+ημ^2(120^0+\alpha)+ημ^2(120-\alpha)\equiv\frac{3}{2}.}$

Λύσις. 1. Ἔχομεν διαδοχικῶς:

$$συν^2\alpha+συν^2(60^0+\alpha)+συν^2(60^0-\alpha)\equiv$$

$$\equiv συν^2\alpha+(συν60^0συν\alpha-ημ60^0ημ\alpha)^2+(συν60^0συν\alpha+ημ60^0ημ\alpha)^2\equiv$$

$$\equiv συν^2\alpha+2συν^2 60^0συν^2\alpha+2ημ^2 60^0ημ^2\alpha\equiv$$

$$\equiv συν^2\alpha+2\cdot\frac{1}{4}συν^2\alpha+2\cdot\frac{3}{4}ημ^2\alpha\equiv\frac{3}{2}\left(συν^2\alpha+ημ^2\alpha\right)=\frac{3}{2}.$$

2. Ἔχομεν διαδοχικῶς:

$$ημ^2\alpha+ημ^2(120^0+\alpha)+ημ^2(120^0-\alpha)\equiv$$

$$\equiv ημ^2\alpha+(ημ120^0συν\alpha+ημ\alpha συν120^0)+(ημ120^0συν\alpha-ημ\alpha συν120^0)^2$$

$$\equiv ημ^2\alpha+2ημ^2 120^0συν^2\alpha+2ημ^2\alpha συν^2 120^0\equiv$$

$$\equiv ημ^2\alpha+2\cdot\frac{3}{4}συν^2\alpha+2ημ^2\alpha\cdot\frac{1}{4}\equiv\frac{3}{2}(ημ^2\alpha+συν^2\alpha)=\frac{3}{2}.$$

17. Νὰ ἀποδειχθῇ ὅτι :

$$\text{συν}^2(\beta-\gamma)+\text{συν}^2(\gamma-\alpha)+\text{συν}^2(\alpha-\beta)-2\text{συν}(\beta-\gamma)\text{συν}(\gamma-\alpha)\,\text{συν}(\alpha-\beta)=1.$$

Λύσις. Θέτομεν $\beta-\gamma=x$, $\gamma-\alpha=y$, $\alpha-\beta=\omega$, ὅτε :

$$x+y+\omega=0 \Rightarrow x+y=-\omega \text{ καὶ } \text{συν}(x+y)=\text{συν}(-\omega)=\text{συν}\omega$$

ἢ $\text{συν}x\text{συν}y-\text{ημ}x\text{ημ}y=\text{συν}\omega$ ἢ $\text{συν}x\text{συν}y-\text{συν}\omega=\text{ημ}x\text{ημ}y$

ἢ $\text{συν}^2x\text{συν}^2y+\text{συν}^2\omega-2\text{συν}x\text{συν}y\text{συν}\omega=\text{ημ}^2x\text{ημ}^2y$

$$=(1-\text{συν}^2x)(1-\text{συν}^2y)=1-\text{συν}^2x-\text{συν}^2y+\text{συν}^2x\text{συν}^2y.$$

ἢ $\text{συν}^2x+\text{συν}^2y+\text{συν}^2\omega-2\text{συν}x\text{συν}y\text{συν}\omega=1$

ἢ $\text{συν}^2(\beta-\gamma)+\text{συν}^2(\gamma-\alpha)+\text{συν}^2(\alpha-\beta)-2\text{συν}(\beta-\gamma)\text{συν}(\gamma-\alpha)\text{συν}(\alpha-\beta)=1.$

18. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $$K=\frac{\alpha^2\text{ημ}(B-\Gamma)}{\text{ημ}B+\text{ημ}\Gamma}+\frac{\beta^2\text{ημ}(\Gamma-A)}{\text{ημ}\Gamma+\text{ημ}A}+\frac{\gamma^2\text{ημ}(A-B)}{\text{ημ}A+\text{ημ}B}=0,$$

2. $$\Lambda=\frac{\alpha^2\text{ημ}(B-\Gamma)}{\text{ημ}A}+\frac{\beta^2\text{ημ}(\Gamma-A)}{\text{ημ}B}+\frac{\gamma^2\text{ημ}(A-B)}{\text{ημ}\Gamma}=0.$$

Λύσις. 1. Τὸ πρῶτον κλάσμα γράφεται διαδοχικῶς :

$$\frac{\alpha^2\text{ημ}(B-\Gamma)}{\text{ημ}B+\text{ημ}\Gamma}=\frac{4R^2\text{ημ}^2A\text{ημ}(B-\Gamma)}{\text{ημ}B+\text{ημ}\Gamma}=\frac{4R^2\text{ημ}A\text{ημ}(B+\Gamma)\text{ημ}(B-\Gamma)}{\text{ημ}B+\text{ημ}\Gamma}=$$

$$=\frac{4R^2\text{ημ}A(\text{ημ}^2B-\text{ημ}^2\Gamma)}{\text{ημ}B+\text{ημ}\Gamma}=4R^2\text{ημ}A(\text{ημ}B-\text{ημ}\Gamma),$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς τῶν Α,Β,Γ, θὰ ἔχωμεν :

$$K=4R^2\text{ημ}A(\text{ημ}B-\text{ημ}\Gamma)+4R^2\text{ημ}B(\text{ημ}\Gamma-\text{ημ}A)+4R^2\text{ημ}\Gamma(\text{ημ}A-\text{ημ}B)=$$

$$=4R^2(\text{ημ}A\text{ημ}B-\text{ημ}A\text{ημ}\Gamma+\text{ημ}B\text{ημ}\Gamma-\text{ημ}A\text{ημ}B+\text{ημ}\Gamma\text{ημ}A-\text{ημ}B\text{ημ}\Gamma)=4R^2\cdot 0=0.$$

2. Εἶναι : $$\frac{\alpha^2\text{ημ}(B-\Gamma)}{\text{ημ}A}=\frac{4R^2\text{ημ}^2A\text{ημ}(B-\Gamma)}{\text{ημ}A}=$$

$$=4R^2\text{ημ}A\text{ημ}(B-\Gamma)=4R^2\text{ημ}(B+\Gamma)\text{ημ}(B-\Gamma)=4R^2(\text{ημ}^2B-\text{ημ}^2\Gamma),$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς θὰ ἔχωμεν :

$$\Lambda=4R^2(\text{ημ}^2B-\text{ημ}^2\Gamma)+4R^2(\text{ημ}^2\Gamma-\text{ημ}^2A)+4R^2(\text{ημ}A-\text{ημ}B)=4R^2\cdot 0=0.$$

19. Ἐὰν $\alpha+\beta+\gamma+\delta=360^0$, νὰ ἀποδειχθῇ ὅτι :

$$\frac{\text{εφ}\alpha+\text{εφ}\beta+\text{εφ}\gamma+\text{εφ}\delta}{\text{σφ}\alpha+\text{σφ}\beta+\text{σφ}\gamma+\text{σφ}\delta}=\text{εφ}\alpha\text{εφ}\beta\text{εφ}\gamma\text{εφ}\delta.$$

Λύσις. Ἔχομεν : $\alpha+\beta=360^0-(\gamma+\delta)$. Ἄρα :

$$\text{εφ}(\alpha+\beta)=\text{εφ}[360^0-(\gamma+\delta)]=-\text{εφ}(\gamma+\delta) \quad \text{ἢ}$$

$$\frac{\text{εφ}\alpha+\text{εφ}\beta}{1-\text{εφ}\alpha\text{εωρ}}=-\frac{\text{εφ}\gamma+\text{εφ}\delta}{1-\text{εφ}\gamma\text{εφ}\delta} \quad \text{ἢ}$$

$$\text{εφ}\alpha+\text{εφ}\beta-\text{εφ}\alpha\text{εφ}\gamma\text{εφ}\delta-\text{εφ}\beta\text{εφ}\gamma\text{εφ}\delta=-\text{εφ}\gamma-\text{εφ}\delta+\text{εφ}\alpha\text{εφ}\beta\text{εφ}\gamma+\text{εφ}\alpha\text{εφ}\beta\text{εφ}\delta$$

ἢ $\text{εφ}\alpha+\text{εφ}\beta+\text{εφ}\gamma+\text{εφ}\delta.=\text{εφ}\alpha\text{εφ}\beta\text{εφ}\gamma\text{εφ}\delta(\text{σφ}\alpha+\text{σφ}\beta+\text{σφ}\gamma+\text{σφ}\delta)$

ἐξ οὗ : $$\frac{\text{εφ}\alpha+\text{εφ}\beta+\text{εφ}\gamma+\text{εφ}\delta}{\text{σφ}\alpha+\text{σφ}\beta+\text{σφ}\gamma+\text{σφ}\delta}=\text{εφ}\alpha\text{εφ}\beta\text{εφ}\gamma\text{εφ}\delta.$$

20. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $$\frac{\text{ημ}(A-B)}{\text{ημ}(A+B)}=\frac{\alpha^2-\beta^2}{\gamma^2}.$$

Λύσις. $\frac{\eta\mu(A-B)}{\eta\mu(A+B)}=\frac{\eta\mu\Gamma\eta\mu(A-B)}{\eta\mu\Gamma\cdot\eta\mu\Gamma}=\frac{\eta\mu(A+B)\eta\mu(A-B)}{\eta\mu^2\Gamma}=$

$$=\frac{\eta\mu^2A-\eta\mu^2B}{\eta\mu^2\Gamma}=\frac{4R^2\eta\mu^2A-4R^2\eta\mu^2B}{4R^2\eta\mu^2\Gamma}=\frac{\alpha^2-\beta^2}{\gamma^2}.$$

2. $$\frac{\boldsymbol{\gamma\eta\mu(A-B)}}{\boldsymbol{\beta\eta\mu(\Gamma-A)}}=\frac{\boldsymbol{\alpha^2-\beta^2}}{\boldsymbol{\gamma^2-\alpha^2}}.$$

Λύσις. Εἶναι : $\frac{\gamma\eta\mu(A-B)}{\beta\eta\mu(\Gamma-A)}=\frac{2R\eta\mu\Gamma\eta\mu(A-B)}{2R\eta\mu B\eta\mu(\Gamma-A)}=$

$$=\frac{\eta\mu(A+B)\eta\mu(A-B)}{\eta\mu(\Gamma+A)\eta\mu(\Gamma-A)}=\frac{\eta\mu^2A-\eta\mu^2B}{\eta\mu^2\Gamma-\eta\mu^2A}=\frac{4R^2\eta\mu^2A-4R^2\eta\mu^2B}{4R^2\eta\mu^2\Gamma-4R^2\eta\mu^2A}=\frac{\alpha^2-\beta^2}{\gamma^2-\alpha^2}.$$

Πρέπει : $\Gamma\neq A$ διὰ νὰ ἔχῃ ἔννοιαν τὸ α΄ μέλος.

3. $$\boldsymbol{K=(\beta+\gamma)\sigma\upsilon\nu A+(\gamma+\alpha)\sigma\upsilon\nu B+(\alpha+\beta)\sigma\upsilon\nu\Gamma=\alpha+\beta+\gamma.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\beta+\gamma)\sigma\upsilon\nu A=(2R\eta\mu B+2R\eta\mu\Gamma)\sigma\upsilon\nu A=2R(\eta\mu B\sigma\upsilon\nu A+\eta\mu\Gamma\sigma\upsilon\nu A)$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς :

$$K=2R(\eta\mu B\sigma\upsilon\nu A+\eta\mu\Gamma\sigma\upsilon\nu A)+2R(\eta\mu\Gamma\sigma\upsilon\nu B+\eta\mu A\sigma\upsilon\nu B)+$$
$$+2R(\eta\mu A\sigma\upsilon\nu\Gamma+\eta\mu B\sigma\upsilon\nu\Gamma)=$$
$$=2R[(\eta\mu A\sigma\upsilon\nu B+\eta\mu B\sigma\upsilon\nu A)+(\eta\mu B\sigma\upsilon\nu\Gamma+\eta\mu\Gamma\sigma\upsilon\nu B)+(\eta\mu\Gamma\sigma\upsilon\nu A+\eta\mu A\sigma\upsilon\nu\Gamma)]$$
$$=2R[\eta\mu(A+B)+\eta\mu(B+\Gamma)+\eta\mu(\Gamma+A)]=2R(\eta\mu\Gamma+\eta\mu A+\eta\mu B)$$
$$=2R\eta\mu A+2R\eta\mu B+2R\eta\mu\Gamma=\alpha+\beta+\gamma.$$

4. $$\boldsymbol{\Lambda\equiv\frac{\alpha-2\gamma\sigma\upsilon\nu B}{\gamma\eta\mu B}+\frac{\beta-2\alpha\sigma\upsilon\nu\Gamma}{\alpha\eta\mu\Gamma}+\frac{\gamma-2\beta\sigma\upsilon\nu A}{\beta\eta\mu A}=0.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{\alpha-2\gamma\sigma\upsilon\nu B}{\gamma\eta\mu B}=\frac{2R\eta\mu A-4R\eta\mu\Gamma\sigma\upsilon\nu B}{2R\eta\mu\Gamma\eta\mu B}=\frac{\eta\mu A-2\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu\Gamma\eta\mu B}=$$

$$\frac{\eta\mu(B+\Gamma)-2\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu\Gamma\eta\mu B}=\frac{\eta\mu B\sigma\upsilon\nu\Gamma+\eta\mu\Gamma\sigma\upsilon\nu B-2\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu\Gamma\eta\mu B}=$$

$$\frac{\eta\mu B\sigma\upsilon\nu\Gamma-\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu\Gamma\eta\mu B}=\frac{\eta\mu B\sigma\upsilon\nu\Gamma}{\eta\mu\Gamma\eta\mu B}-\frac{\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu\Gamma\eta\mu B}=\sigma\varphi\Gamma-\sigma\varphi B,$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς τῶν A,B,Γ θὰ ἔχωμεν :

$$\Lambda\equiv\sigma\varphi\Gamma-\sigma\varphi B+\sigma\varphi A-\sigma\varphi\Gamma+\sigma\varphi B-\sigma\varphi A=0.$$

21. Ἐὰν $\boldsymbol{\alpha+\beta+\gamma=\frac{\pi}{2}}$, **νὰ ἀποδειχθῇ ὅτι :**

$$\boldsymbol{\sigma\varphi\alpha+\sigma\varphi\beta+\sigma\varphi\gamma=\sigma\varphi\alpha\sigma\varphi\beta\sigma\varphi\gamma.}$$

Λύσις Ἔχομεν $\alpha+\beta=\frac{\pi}{2}-\gamma$ ἢ $\sigma\varphi(\alpha+\beta)=\sigma\varphi\left(\frac{\pi}{2}-\gamma\right)=\frac{1}{\sigma\varphi\gamma}$

ἢ $\frac{\sigma\varphi\alpha\sigma\varphi\beta-1}{\sigma\varphi\alpha+\sigma\varphi\beta}=\frac{1}{\sigma\varphi\gamma}$ ἢ $\sigma\varphi\alpha\sigma\varphi\beta\sigma\varphi\gamma-\sigma\varphi\gamma=\sigma\varphi\alpha+\sigma\varphi\beta$

ἐξ οὗ : $$\sigma\varphi\alpha+\sigma\varphi\beta+\sigma\varphi\gamma=\sigma\varphi\alpha\cdot\sigma\varphi\beta\cdot\sigma\varphi\gamma.$$

Ἀντιστρόφως, ἂν ἰσχύῃ ἡ (1), πῶς συνδέονται αἱ γωνίαι α,β,γ;

Ἔχομεν : $$(\sigma\varphi\alpha\sigma\varphi\beta-1)\sigma\varphi\gamma=\sigma\varphi\alpha+\sigma\varphi\beta$$

ἢ $\frac{\sigma\varphi\alpha\sigma\varphi\beta-1}{\sigma\varphi\alpha+\sigma\varphi\beta}=\frac{1}{\sigma\varphi\gamma}$ ἢ $\sigma\varphi(\alpha+\beta)=\frac{1}{\sigma\varphi\gamma}=\sigma\varphi\left(\frac{\pi}{2}-\gamma\right)$

ἐξ οὗ: $\alpha+\beta=\frac{\pi}{2}-\gamma+k\pi$ ἢ $\boxed{\alpha+\beta+\gamma=k\pi+\frac{\pi}{2}}$, $k\in\mathbf{Z}$.

22. Ἐὰν $x>0$, $0<\alpha<\frac{\pi}{2}$, $0<\beta<\frac{\pi}{2}$, $0<\gamma<\frac{\pi}{2}$ καὶ

$\sigma\varphi\alpha=\sqrt{x^3+x^2+x}$, $\sigma\varphi\beta=\sqrt{x+x^{-1}+1}$, $\sigma\varphi\gamma=\sqrt{x^{-3}+x^{-2}+x^{-1}}$, νὰ ἀποδειχθῇ ὅτι $\alpha+\beta=\gamma$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\varphi(\alpha+\beta)=\frac{\sigma\varphi\alpha\sigma\varphi\beta-1}{\sigma\varphi\alpha+\sigma\varphi\beta}=\frac{\sqrt{x^3+x^2+x}\cdot\sqrt{x^3+x^{-1}+1}-1}{\sqrt{x^3+x^2+x}+\sqrt{x+x^{-1}+1}}=\frac{x\sqrt{x}}{\sqrt{x^2+x+1}}$$

$$=\sqrt{x^{-3}+x^{-2}+x^{-1}}=\sigma\varphi\gamma \qquad \text{Ἄρα} \quad \alpha+\beta=\gamma.$$

23. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

$$\eta\mu A\eta\mu(B-\Gamma)+\eta\mu B\eta\mu(\Gamma-A)+\eta\mu\Gamma\eta\mu(A-B)=0.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu A\eta\mu(B-\Gamma)=\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)=\eta\mu^2B-\eta\mu^2\Gamma$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς τῶν Α,Β,Γ λαμβάνομεν:

$$\Sigma\eta\mu A\eta\mu(B-\Gamma)=\eta\mu^2B-\eta\mu^2\Gamma+\eta\mu^2\Gamma-\eta\mu^2A+\eta\mu^2A-\eta\mu^2B=0.$$

24. Ἐὰν $\alpha+\beta+\gamma=\frac{\pi}{2}$, νὰ ἀποδειχθῇ ὅτι.

1. **$\eta\mu^2\alpha+\eta\mu^2\beta+\eta\mu^2\gamma+2\eta\mu\alpha\eta\mu\beta\eta\mu\gamma=1$** (1)
2. **Πῶς συνδέονται αἱ γωνίαι α, β, γ, ἂν ἰσχύῃ ἡ (1);**

Λύσις. 1. Ἐκ τῆς $\alpha+\beta+\gamma=\frac{\pi}{2}\Longrightarrow\alpha+\beta=\frac{\pi}{2}-\gamma$

ἢ $\sigma\upsilon\nu(\alpha+\beta)=\sigma\upsilon\nu\left(\frac{\pi}{2}-\gamma\right)=\eta\mu\gamma$ ἢ $\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta-\eta\mu\alpha\eta\mu\beta=\eta\mu\gamma$

ἢ $$\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta=\eta\mu\alpha\eta\mu\beta+\eta\mu\gamma$$

ἢ $$\sigma\upsilon\nu^2\alpha\sigma\upsilon\nu^2\beta=\eta\mu^2\alpha\eta\mu^2\beta+\eta\mu^2\gamma+2\eta\mu\alpha\eta\mu\beta\eta\mu\gamma$$

ἢ $$(1-\eta\mu^2\alpha)(1-\eta\mu^2\beta)=\eta\mu^2\alpha\eta\mu^2\beta+\eta\mu^2\gamma+2\eta\mu\alpha\eta\mu\beta\eta\mu\gamma$$

καὶ μετὰ τὰς πράξεις, λαμβάνομεν:

$$\eta\mu^2\alpha+\eta\mu^2\beta+\eta\mu^2\gamma+2\eta\mu\alpha\eta\mu\beta\eta\mu\gamma=1 \qquad (2)$$

2. Ἀκολουθοῦντες ἀντίστροφον πορείαν, γράφομεν τὴν (2) ὑπὸ τὴν μορφήν:

$$\sigma\upsilon\nu^2\alpha\sigma\upsilon\nu^2\beta=(\eta\mu\alpha\eta\mu\beta+\eta\mu\gamma)^2$$

ἢ $$(\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu\alpha\eta\mu\beta+\eta\mu\gamma)(\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta-\eta\mu\alpha\eta\mu\beta-\eta\mu\gamma)=0$$

ἢ $$[\sigma\upsilon\nu(\alpha-\beta)+\eta\mu\gamma][\sigma\upsilon\nu(\alpha+\beta)-\eta\mu\gamma]=0, \quad \text{ὁπότε}$$

ἢ $\sigma\upsilon\nu(\alpha-\beta)+\eta\mu\gamma=0$ ἢ $\sigma\upsilon\nu(\alpha-\beta)=-\eta\mu\gamma=\eta\mu(-\gamma)=\sigma\upsilon\nu\left(\frac{\pi}{2}+\gamma\right)$

ἐξ οὗ: $\alpha-\beta=2k\pi\pm\left(\frac{\pi}{2}+\gamma\right)$ ἢ $\boxed{\alpha-\beta\mp\gamma=2k\pi+\frac{\pi}{2}}$, $k\in\mathbf{Z}$

Ἢ $\quad συν(α+β)-ημγ=0 \quad$ ἢ $\quad συν(α+β)=ημγ=συν\left(\frac{π}{2}-γ\right),$

ἐξ οὗ: $\quad α+β=2k_1π\pm\left(\frac{π}{2}-γ\right)$ ἢ $\boxed{α+β\pm γ=2k_1π\pm\frac{π}{2}}$, $k_1\in \mathbf{Z}$

25. Ἐὰν $A+B=225^0$, νὰ ἀποδειχθῇ ὅτι:

$$\frac{σφA}{1+σφA}\cdot\frac{σφB}{1+σφB}=\frac{1}{2}$$

Λύσις. Ἔχομεν: $\quad σφ(A+B)=σφ225^0=σφ(180^0+45^0)=σφ45^0=1$

ἢ $$\frac{σφAσφB-1}{σφA+σφB}=1 \quad ἢ \quad σφAσφB-1=σφA+σφB$$

ἢ $$2σφAσφB=1+σφA+σφB+σφAσφB=(1+σφA)(1+σφB),$$

ἐξ οὗ $$\frac{σφA}{1+σφA}\cdot\frac{σφB}{1+σφB}=\frac{1}{2}.$$

26. Ἐὰν $α+β+γ=\frac{π}{2}$, νὰ ἀποδειχθῇ ὅτι:

$$εφ^2α+εφ^2β+εφ^2γ\geqslant 1.$$

Λύσις. Εἶναι: $$\left.\begin{array}{l}(εφα-εφβ)^2\geqslant 0\\(εφβ-εφγ)^2\geqslant 0\\(εφγ-εφα)^2\geqslant 0\end{array}\right\}\Longrightarrow\left.\begin{array}{l}εφ^2α+εφ^2β\geqslant 2εφαεφβ\\εφ^2β+εφ^2γ\geqslant 2εφβεφγ\\εφ^2γ+εφ^2α\geqslant 2εφγεφα\end{array}\right\}$$

ἐξ ὧν, διὰ προσθέσεως κατὰ μέλη, λαμβάνομεν:

$$εφ^2α+εφ^2β+εφ^2γ\geqslant εφαεφβ+εφβεφγ+εφγεφα. \qquad (1)$$

Ἐκ τῆς $\quad α+β+γ=\frac{π}{2}\Longrightarrow α+β=\frac{π}{2}-γ \quad$ ἢ

$$εφ(α+β) \quad εφ\left(\frac{π}{2}-γ\right)=σφγ=\frac{1}{εφγ} \quad ἢ \quad \frac{εφα+εφβ}{1-εφαεφβ}=\frac{1}{εφγ}$$

ἐξ οὗ: $$εφαεφβ+εφβεφγ+εφγεφα=1$$

καὶ ἡ (1) γίνεται: $$εφ^3α+εφ^2β+εφ^2γ\geqslant 1.$$

27. Ἐὰν $α+β+γ=π$, νὰ ἀποδειχθῇ ὅτι:

1. $\quad σφ^2α+σφ^2β+σφ^2γ\geqslant 1,$

2. $\quad εφ^2\frac{α}{2}+εφ^2\frac{β}{2}+εφ^2\frac{γ}{2}\geqslant 1.$

Λύσις. 1. Ἔχομεν: $$\left.\begin{array}{l}(σφα-σφβ)^2\geqslant 0\\(σφβ-σφγ)^2\geqslant 0\\(σφγ-σφα)^2\geqslant 0\end{array}\right\}\Longrightarrow\left.\begin{array}{l}σφ^2α+σφ^2β\geqslant 2σφασφβ\\σφ^2β+σφ^2γ\geqslant 2σφβσφγ\\σφ^2γ+σφ^2α\geqslant 2σφγσφα\end{array}\right\}$$

ἐξ οὗ: $$σφ^2α+σφ^2β+σφ^2γ\geqslant σφασφβ+σφβσφγ+σφγσφα. \qquad (1)$$

Ἀλλὰ (ἄσκ. 15,2) εἶναι: $σφασφβ+σφβσφγ+σφγσφα=1$ καὶ ἡ (1) γίνεται: $\quad σφ^2α+σφ^2β+σφ^2γ\geqslant 1.$

2. Είναι :
$$\left.\begin{aligned}\left(\varepsilon\varphi\frac{\alpha}{2}-\varepsilon\varphi\frac{\beta}{2}\right)^2\geqslant 0\\ \left(\varepsilon\varphi\frac{\beta}{2}-\varepsilon\varphi\frac{\gamma}{2}\right)^2\geqslant 0\\ \left(\varepsilon\varphi\frac{\gamma}{2}-\varepsilon\varphi\frac{\alpha}{2}\right)^2\geqslant 0\end{aligned}\right\} \Rightarrow \left.\begin{aligned}\varepsilon\varphi^2\frac{\alpha}{2}+\varepsilon\varphi^2\frac{\beta}{2}\geqslant 2\varepsilon\varphi\frac{\alpha}{2}\,\varepsilon\varphi\frac{\beta}{2}\\ \varepsilon\varphi^2\frac{\beta}{2}+\varepsilon\varphi^2\frac{\gamma}{2}\geqslant 2\varepsilon\varphi\frac{\beta}{2}\,\varepsilon\varphi\frac{\gamma}{2}\\ \varepsilon\varphi^2\frac{\gamma}{2}+\varepsilon\varphi^2\frac{\alpha}{2}\geqslant 2\varepsilon\varphi\frac{\gamma}{2}\,\varepsilon\varphi\frac{\alpha}{2}\end{aligned}\right\}$$

ἐξ οὗ : $\varepsilon\varphi^2\frac{\alpha}{2}+\varepsilon\varphi^2\frac{\beta}{2}+\varepsilon\varphi^2\frac{\gamma}{2}\geqslant\varepsilon\varphi\frac{\alpha}{2}\,\varepsilon\varphi\frac{\beta}{2}+\varepsilon\varphi\frac{\beta}{2}\,\varepsilon\varphi\frac{\gamma}{2}+\varepsilon\varphi\frac{\gamma}{2}\varepsilon\varphi\frac{\alpha}{2}$ (2)

Ἀλλὰ $\alpha+\beta+\gamma=\pi \Rightarrow \frac{\alpha}{2}+\frac{\beta}{2}=\frac{\pi}{2}-\frac{\gamma}{2}$ ἤ

$$\varepsilon\varphi\left(\frac{\alpha}{2}+\frac{\beta}{2}\right)=\varepsilon\varphi\left(\frac{\pi}{2}-\frac{\gamma}{2}\right)=\sigma\varphi\frac{\gamma}{2}=\frac{1}{\varepsilon\varphi\frac{\gamma}{2}} \text{ ἤ } \frac{\varepsilon\varphi\frac{\alpha}{2}+\varepsilon\varphi\frac{\beta}{2}}{1-\varepsilon\varphi\frac{\alpha}{2}\,\varepsilon\varphi\frac{\beta}{2}}=\frac{1}{\varepsilon\varphi\frac{\gamma}{2}}$$

ἐξ οὗ : $\varepsilon\varphi\frac{\alpha}{2}\,\varepsilon\varphi\frac{\beta}{2}+\varepsilon\varphi\frac{\beta}{2}\,\varepsilon\varphi\frac{\gamma}{2}+\varepsilon\varphi\frac{\gamma}{2}\,\varepsilon\varphi\frac{\alpha}{2}=1$

καὶ ἡ (2) γίνεται :

$$\varepsilon\varphi^2\frac{\alpha}{2}+\varepsilon\varphi^2\frac{\beta}{2}+\varepsilon\varphi^2\frac{\gamma}{2}\geqslant 1.$$

28. Ἐὰν $0<x<\frac{\pi}{2}$, $0<y<\frac{\pi}{2}$, νὰ ἀποδειχθῇ ὅτι :

$$\eta\mu(x+y)<\eta\mu x+\eta\mu y.$$

Λύσις. Ἐπειδὴ
$$\left.\begin{aligned}0<x<\frac{\pi}{2} &\Rightarrow \sigma\upsilon\nu x<1\\ 0<y<\frac{\pi}{2} &\Rightarrow \sigma\upsilon\nu y<1\end{aligned}\right\}$$

καὶ ἐπειδὴ $\eta\mu y>0$, $\eta\mu x>0$, θὰ εἶναι καὶ

$$\left.\begin{aligned}\sigma\upsilon\nu x\,\eta\mu y<\eta\mu y\\ \sigma\upsilon\nu y\,\eta\mu x<\eta\mu x\end{aligned}\right\} \Rightarrow \eta\mu x\,\sigma\upsilon\nu y+\sigma\upsilon\nu x\,\eta\mu y<\eta\mu x+\eta\mu y$$

ἤ $\eta\mu(x+y)<\eta\mu x+\eta\mu y$,

29. Ἐὰν αἱ γωνίαι τοῦ τριγώνου ΑΒΓ ἐπαληθεύουν τὴν ἰσότητα :

$$\eta\mu^2 A+\eta\mu^2 B+\eta\mu^2 \Gamma=2,$$

νὰ ἀποδειχθῇ ὅτι τὸ τρίγωνον ΑΒΓ εἶναι ὀρθογώνιον.

Λύσις. Εἶναι $A+B+\Gamma=\pi$. Ἄρα (ἄσκ. 15, 6) εἶναι :

$$\eta\mu^2 A+\eta\mu^2 B+\eta\mu^2\Gamma-2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma=2,$$

ἐξ οὗ : $\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma=0,$

ὁπότε ἢ $\sigma\upsilon\nu A=0$, ἐξ οὗ $A=90^0$
ἢ $\sigma\upsilon\nu B=0$, » $B=90^0$
ἢ $\sigma\upsilon\nu\Gamma=0$, » $\Gamma=90^0$

Ἄρα τὸ τρίγωνον ΑΒΓ θὰ εἶναι ὀρθογώνιον.

30. Ἐὰν $\dfrac{\varepsilon\varphi(\alpha-\beta)}{\varepsilon\varphi\alpha}+\dfrac{\eta\mu^2\gamma}{\eta\mu^2\alpha}=1$, **νὰ δειχθῇ ὅτι:** $\varepsilon\varphi^2\gamma=\varepsilon\varphi\alpha\varepsilon\varphi\beta$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται διαδοχικῶς:

$$\frac{\eta\mu^2\gamma}{\eta\mu^2\alpha}=1-\frac{\varepsilon\varphi(\alpha-\beta)}{\varepsilon\varphi\alpha}=1-\frac{\varepsilon\varphi\alpha-\varepsilon\varphi\beta}{(1+\varepsilon\varphi\alpha\varepsilon\varphi\beta)\varepsilon\varphi\alpha}=\frac{\varepsilon\varphi\alpha+\varepsilon\varphi^2\alpha\varepsilon\varphi\beta-\varepsilon\varphi\alpha+\varepsilon\varphi\beta}{(1+\varepsilon\varphi\alpha\varepsilon\varphi\beta)\varepsilon\varphi\alpha}=$$

$$=\frac{\varepsilon\varphi^2\alpha\varepsilon\varphi\beta+\varepsilon\varphi\beta}{(1+\varepsilon\varphi\alpha\varepsilon\varphi\beta)\varepsilon\varphi\alpha}$$

ἢ $$\frac{\dfrac{\varepsilon\varphi^2\gamma}{1+\varepsilon\varphi^2\gamma}}{\dfrac{\varepsilon\varphi^2\alpha}{1+\varepsilon\varphi^2\alpha}}=\frac{\varepsilon\varphi^2\alpha\varepsilon\varphi\beta+\varepsilon\varphi\beta}{(1+\varepsilon\varphi\alpha\varepsilon\varphi\beta)\varepsilon\varphi\alpha}$$ ἢ $$\frac{\varepsilon\varphi^2\gamma(1+\varepsilon\varphi^2\alpha)}{\varepsilon\varphi^2\alpha(1+\varepsilon\varphi^2\gamma)}=\frac{\varepsilon\varphi\beta(1+\varepsilon\varphi^2\alpha)}{(1+\varepsilon\varphi\alpha\varepsilon\varphi\beta)\varepsilon\varphi\alpha}$$

ἢ $$\frac{\varepsilon\varphi^2\gamma}{\varepsilon\varphi\alpha(1+\varepsilon\varphi^2\gamma)}=\frac{\varepsilon\varphi\beta}{1+\varepsilon\varphi\alpha\varepsilon\varphi\beta}$$ ἢ $$\varepsilon\varphi^2\gamma+\varepsilon\varphi^2\gamma\varepsilon\varphi\alpha\varepsilon\varphi\beta=\varepsilon\varphi\alpha\varepsilon\varphi\beta(1+\varepsilon\varphi^2\gamma)$$

ἢ $$\varepsilon\varphi^2\gamma(1+\varepsilon\varphi\alpha\varepsilon\varphi\beta)=\varepsilon\varphi\alpha\varepsilon\varphi\beta(1+\varepsilon\varphi^2\gamma)$$

ἐξ οὗ: $$\varepsilon\varphi^2\gamma=\varepsilon\varphi\alpha\varepsilon\varphi\beta.$$

31. Νὰ ἀποδειχθῇ ὅτι:

1. $\eta\mu x+\sigma\upsilon\nu x=\sqrt{2}\,\sigma\upsilon\nu\left(\dfrac{\pi}{4}-x\right)=\sqrt{2}\,\eta\mu\left(\dfrac{\pi}{4}+x\right)$,

2. $\sigma\upsilon\nu x-\eta\mu x=\sqrt{2}\,\eta\mu\left(\dfrac{\pi}{4}-x\right)=\sqrt{2}\,\sigma\upsilon\nu\left(\dfrac{\pi}{4}+x\right)$.

Λύσις. 1. Ἔχομεν διαδοχικῶς:

$$\eta\mu x+\sigma\upsilon\nu x=\sqrt{2}\left(\frac{\sqrt{2}}{2}\eta\mu x+\frac{\sqrt{2}}{2}\sigma\upsilon\nu x\right)=\sqrt{2}\left(\eta\mu\frac{\pi}{4}\eta\mu x+\sigma\upsilon\nu\frac{\pi}{4}\sigma\upsilon\nu x\right)=$$

$$=\sqrt{2}\,\sigma\upsilon\nu\left(\frac{\pi}{4}-x\right)=\sqrt{2}\,\eta\mu\left(\frac{\pi}{4}+x\right).$$

2. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu x-\eta\mu x=\sqrt{2}\left(\frac{\sqrt{2}}{2}\sigma\upsilon\nu x-\frac{\sqrt{2}}{2}\eta\mu x\right)=\sqrt{2}\left(\sigma\upsilon\nu\frac{\pi}{4}\sigma\upsilon\nu x-\eta\mu\frac{\pi}{4}\eta\mu x\right)=$$

$$=\sqrt{2}\,\sigma\upsilon\nu\left(\frac{\pi}{4}+x\right)=\sqrt{2}\,\eta\mu\left(\frac{\pi}{4}-x\right).$$

32. Νὰ ὑπολογισθοῦν οἱ ἀκόλουθοι τριγωνομετρικοὶ ἀριθμοί:

1. $\eta\mu(\beta+\gamma-\alpha)$, $\eta\mu(\gamma+\alpha-\beta)$, $\eta\mu(\alpha+\beta-\gamma)$.

Λύσις. Γνωρίζομεν ὅτι:

$$\eta\mu(\alpha+\beta+\gamma)=\eta\mu\alpha\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma+\eta\mu\beta\sigma\upsilon\nu\alpha\sigma\upsilon\nu\gamma+\eta\mu\gamma\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta-\eta\mu\alpha\eta\mu\beta\eta\mu\gamma. \quad (1)$$

Θέτομεν ὅπου α τὸ —α καὶ ἔχομεν:

$$\eta\mu(\beta+\gamma-\alpha)=\eta\mu(-\alpha)\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma+\eta\mu\beta\sigma\upsilon\nu(-\alpha)\sigma\upsilon\nu\gamma+\eta\mu\gamma\sigma\upsilon\nu(-\alpha)\sigma\upsilon\nu\beta-$$

$$-\eta\mu(-\alpha)\eta\mu\beta\eta\mu\gamma=-\eta\mu\alpha\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma+\eta\mu\beta\sigma\upsilon\nu\alpha\sigma\upsilon\nu\gamma+\eta\mu\gamma\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu\alpha\eta\mu\beta\eta\mu\gamma.$$

Ἐὰν εἰς τὴν σχέσιν (1) θέσωμεν ὅπου β τὸ $-\beta$, λαμβάνομεν:

$$ημ(γ+α-β) = ημασυν(-β)συνγ + ημ(-β)συνασυνγ + ημγσυνασυν(-β) -$$
$$-ημαημ(-β)ημγ = ημασυνβσυνγ - ημβσυνασυνγ + ημγσυνασυνβ + ημαημβημγ.$$

Εἰς τήν σχέσιν (1) θέτομεν ὅπου γ τὸ $-\gamma$ καὶ λαμβάνομεν:

$$ημ(α+β-γ) = ημασυνβσυν(-γ) + ημβσυνασυν(-γ) + ημ(-γ)συνασυνβ -$$
$$-ημαημβημ(-γ) = ημασυνβσυνγ + ημβσυνασυνγ - ημγσυνασυνβ + ημαημβημγ.$$

2. Ἐὰν εἰς τὴν σχέσιν (1) θέσωμεν ὅπου β τὸ $-\beta$ καὶ ὅπου γ τὸ $-\gamma$, λαμβάνομεν:

$$ημ(α-β-γ) = ημασυν(-β)συν(-γ) + ημ(-β)συνασυν(-γ) +$$
$$+ημ(-γ)συνασυν(-β) - ημαημ(-β)ημ(-γ) =$$
$$= ημασυνβσυνγ - ημβσυνασυνγ - ημγσυνασυνβ - ημαημβημγ.$$

Ὁμοίως, ἂν θέσωμεν ὅπου α τὸ $-\alpha$ καὶ ὅπου γ τὸ $-\gamma$, ἔχομεν:

$$ημ(β-α-γ) = ημ(-α)συνβσυν(-γ) + ημβσυν(-α)συν(-γ) +$$
$$+ημ(-γ)συν(-α)συνβ - ημ(-α)ημβημ(-γ) =$$
$$= -ημασυνβσυνγ + ημβσυνασυνγ - ημγσυνασυνβ - ημαημβημγ.$$

Ὁμοίως, ἂν θέσωμεν ὅπου α τὸ $-\alpha$ καὶ ὅπου β τὸ $-\beta$, ἔχομεν:

$$ημ(γ-α-β) = ημ(-α)συν(-β)συνγ + ημ(-β)συν(-α)συνγ + ημγσυν(-α)συν(-β)$$
$$-ημ(-α)ημ(-β)ημγ =$$
$$= -ημασυνβσυνγ - ημβσυνασυνγ + ημγσυνασυνβ - ημαημβημγ.$$

Ὁμοίως ἐργαζόμενοι, εὑρίσκομεν τὰ ἐξαγόμενα τῶν 3, 4, 5, 6, 7, 8 ἔχοντες ὑπ' ὄψιν τοὺς τύπους 16—17—18.

33. Ἐὰν $εφα = \frac{3}{4}$, $εφβ = \frac{8}{15}$, $εφγ = \frac{5}{12}$ καὶ $0 < (α, β, γ) < \frac{\pi}{2}$, νὰ ὑπολογισθοῦν οἱ τριγωνομετρικοὶ ἀριθμοὶ τῶν ἀθροισμάτων $α \pm β \pm γ$.

Λύσις. Ἔχομεν:

$$ημα = \frac{εφα}{\sqrt{1+εφ^2α}} = \frac{\frac{3}{4}}{\sqrt{1+\frac{9}{16}}} = \frac{3}{5}, \quad καὶ\ συνα = \frac{1}{\sqrt{1+εφ^2α}} = \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{4}{5}.$$

$$ημβ = \frac{εφβ}{\sqrt{1+εφ^2β}} = \frac{\frac{8}{15}}{\sqrt{1+\frac{64}{225}}} = \frac{8}{17}, \quad καὶ\ συνβ = \frac{1}{\sqrt{1+εφ^2β}} = \frac{15}{17}.$$

$$ημγ = \frac{εφγ}{\sqrt{1+εφ^2γ}} = \frac{\frac{5}{12}}{\sqrt{1+\frac{25}{144}}} = \frac{5}{13}, \quad καὶ\ συνγ = \frac{1}{\sqrt{1+\frac{25}{144}}} = \frac{12}{13}.$$

Εἶναι δὲ καὶ $σφα = \frac{4}{3}$, $σφβ = \frac{15}{8}$, $σφγ = \frac{12}{5}$.

Ἀκολούθως ἐργαζόμεθα κατὰ τὴν ἄσκησιν 32.

34. Ἐὰν $ημα = \frac{3}{5}$, $ημβ = \frac{12}{13}$, $ημγ = \frac{7}{25}$, νὰ ὑπολογισθοῦν τὰ $ημ(α+β+γ)$, $εφ(α+β+γ)$, δεδομένου ὅτι $0 < (α,β,γ) < \frac{π}{2}$.

Λύσις. Θὰ εἶναι:

$$συνα = \sqrt{1-ημ^2α} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}, \text{ καὶ } εφα = \frac{3}{4} \Rightarrow σφα = \frac{4}{3}$$

$$συνβ = \sqrt{1-ημ^2β} = \sqrt{1-\frac{144}{169}} = \frac{5}{13}, \text{ καὶ } εφβ = \frac{12}{5} \Rightarrow σφβ = \frac{5}{12}$$

$$συνγ = \sqrt{1-ημ^2γ} = \sqrt{1-\frac{49}{625}} = \frac{24}{25}, \text{ καὶ } εφγ = \frac{7}{24} \Rightarrow σφγ = \frac{24}{7}.$$

Ἤδη, εὐκόλως εὑρίσκομεν τὰ ζητούμενα, βάσει τῶν τύπων 15 καὶ 17.

35. Ἐὰν $ημα = 0,4$ καὶ $90^0 < α < 180^0$, νὰ ὑπολογισθοῦν οἱ ἀριθμοί: $ημ2α$, $συν2α$, $εφ2α$, $σφ2α$.

Λύσις. Εἶναι $ημα = 0,4 = \frac{4}{10} = \frac{2}{5}$ καὶ ἄρα:

$$συνα = -\sqrt{1-ημ^2α} = -\sqrt{1-\frac{4}{25}} = -\frac{\sqrt{21}}{5}, \text{ ὅτε } εφα = -\frac{2}{\sqrt{21}} = -\frac{2\sqrt{21}}{21}$$

καὶ $σφα = -\frac{\sqrt{21}}{2}$. Κατ' ἀκολουθίαν:

$$ημ2α = 2ημασυνα = 2 \cdot \frac{2}{5} \cdot \left(-\frac{\sqrt{21}}{5}\right) = -\frac{4\sqrt{21}}{25}$$

$$συν2α = 2συν^2α - 1 = 2 \cdot \frac{21}{25} - 1 = \frac{42-25}{25} = \frac{17}{15}$$

$$εφ2α = \frac{ημ2α}{συν2α} = \frac{\frac{-4\sqrt{21}}{25}}{\frac{17}{25}} = -\frac{4\sqrt{21}}{17}, \text{ καὶ ἄρα } σφ2α = -\frac{17}{4\sqrt{21}}.$$

36. Ἐὰν $συνα = \frac{1}{3}$ καὶ $0^0 < α < 90^0$, νὰ ὑπολογισθοῦν οἱ ἀριθμοί: $ημ2α$, $συν2α$, $εφ2α$, $σφ2α$

Λύσις. Θὰ ἔχωμεν:

$$ημα = \sqrt{1-συν^2α} = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}, \text{ ὅτε } εφα = 2\sqrt{2},$$

$$σφα = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}, \text{ καὶ } ημ2α = 2ημασυνα = 2 \cdot \frac{2\sqrt{2}}{3} \cdot \frac{1}{3} = \frac{4\sqrt{2}}{9}$$

$$συν2α = 2συν^2α - 1 = 2 \cdot \frac{1}{9} - 1 = \frac{2-9}{9} = \frac{-7}{9}$$

$$\varepsilon\varphi 2\alpha = \frac{\eta\mu 2\alpha}{\sigma\upsilon\nu 2\alpha} = \frac{\frac{4\sqrt{2}}{9}}{-\frac{7}{9}} = -\frac{4\sqrt{2}}{7}, \text{ καὶ } \sigma\varphi 2\alpha = -\frac{7}{4\sqrt{2}} = -\frac{7\sqrt{2}}{8}.$$

37. Ἐὰν $\eta\mu x - \sigma\upsilon\nu x = 0{,}2$, νὰ ὑπολογισθῇ τὸ $\eta\mu 2x$.

Λύσις. Ἐκ τῆς $\eta\mu x - \sigma\upsilon\nu x = 0{,}2 = \frac{2}{10} = \frac{1}{5}$ ἔχομεν:

$$\eta\mu^2 x + \sigma\upsilon\nu^2 x - 2\eta\mu x\sigma\upsilon\nu x = \frac{1}{25}$$

ἢ

$$1 - \eta\mu 2x = \frac{1}{25} \Rightarrow \eta\mu 2x = \frac{24}{25}.$$

38. Ἐὰν $\eta\mu\alpha = \frac{1}{3}$, $\eta\mu\beta = \frac{1}{2}$ καὶ $0 < (\alpha, \beta) < \frac{\pi}{2}$, νὰ ὑπολογισθῇ τὸ $\eta\mu(2\alpha+\beta)$.

Λύσις. Εἶναι $\sigma\upsilon\nu\alpha = \sqrt{1-\eta\mu^2\alpha} = \sqrt{1-\frac{1}{9}} = \frac{2\sqrt{2}}{3}$

καὶ $\sigma\upsilon\nu\beta = \sqrt{1-\eta\mu^2\beta} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$. Ἄρα:

$$\eta\mu(2\alpha+\beta) = \eta\mu 2\alpha\sigma\upsilon\nu\beta + \eta\mu\beta\sigma\upsilon\nu 2\alpha = 2\eta\mu\alpha\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta + \eta\mu\beta\,(2\sigma\upsilon\nu^2\alpha - 1)$$

$$= 2\cdot\frac{1}{3}\cdot\frac{2\sqrt{2}}{3}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\left(2\cdot\frac{8}{9} - 1\right)$$

$$= \frac{2\sqrt{6}}{9} + \frac{7}{18} = \frac{4\sqrt{6}+7}{18}.$$

39. Ἐὰν $4\eta\mu^2 x - 2(1+\sqrt{3})\,\eta\mu x + \sqrt{3} = 0$ νὰ ὑπολογισθοῦν οἱ ἀριθμοὶ $\eta\mu 2x$, $\sigma\upsilon\nu 2x$, $\varepsilon\varphi 2x$.

Λύσις. Ἡ δοθεῖσα ἰσότης εἶναι ἐξίσωσις β′ βαθμοῦ ὡς πρὸς ημx. Ἄρα:

$$\eta\mu x = \frac{2(1+\sqrt{3}) \pm \sqrt{4(1+\sqrt{3})^2 - 16\sqrt{3}}}{8} = \frac{2(1+\sqrt{3}) \pm \sqrt{4(4+2\sqrt{3}) - 16\sqrt{3}}}{8} =$$

$$= \frac{2(1+\sqrt{3}) \pm \sqrt{16+8\sqrt{3}-16\sqrt{3}}}{8} = \frac{1+\sqrt{3} \pm \sqrt{4-2\sqrt{3}}}{4} = \frac{1+\sqrt{3} \pm \sqrt{(1-\sqrt{3})^2}}{4}$$

$$= \frac{1+\sqrt{3} \pm (1-\sqrt{3})}{4}$$

ἐξ οὗ $\eta\mu x = \frac{1+\sqrt{3}+1-\sqrt{3}}{4} = \frac{1}{2}$ καὶ $\eta\mu x = \frac{1+\sqrt{3}-1+\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$

καὶ $\quad συνx = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$ καὶ $συνx = \sqrt{1-\frac{3}{4}} = \frac{1}{2}$.

Ἄρα : $\quad ημ2x = 2ημxσυνx = 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$

$$ημ2x = 2ημxσυνx = 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$$

$$συν2x = 2συν^2x - 1 = 2 \cdot \frac{3}{4} - 1 = \frac{1}{2}$$

$$συν2x = 2συν^2x - 1 = 2 \cdot \frac{1}{4} - 1 = -\frac{1}{2}$$

$$εφ2x = \frac{ημ2x}{συν2x} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}, \quad σφ2x = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \text{ κλπ.}$$

40. Ἐὰν $συνα = \frac{1}{3}$, νὰ ὑπολογισθῇ τὸ $συν3α$.

Λύσις. Εἶναι : $\quad συν3α = 4συν^3α - 3συνα = 4 \cdot \frac{1}{27} - 1 \cdot \frac{1}{3} =$

$$= \frac{4}{27} - 1 = \frac{4-27}{27} = -\frac{23}{27}.$$

41. Ἐὰν $ημα = \frac{3}{5}$. νὰ ὑπολογισθῇ τὸ $ημ3α$.

Λύσις. Εἶναι : $\quad ημ3α = 3ημα - 4ημ^3α = 3 \cdot \frac{3}{5} - 4 \cdot \frac{27}{125} =$

$$= \frac{9}{5} - \frac{108}{125} = \frac{225-108}{125} = \frac{117}{125}.$$

42. Ἐὰν $εφα = 3$, νὰ ὑπολογισθῇ ἡ $εφ3α$.

Λύσις. Ἔχομεν διαδοχικῶς :

$$εφ3α = \frac{3εφα - εφ^3α}{1-3εφ^2α} = \frac{3 \cdot 3 - 3^3}{1-3 \cdot 3^2} = \frac{9-27}{1-27} = \frac{-18}{-26} = \frac{9}{13}.$$

43. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ἰσότητες :

1. $\quad \dfrac{ημ2α}{1+συν2α} = εφα.$

Λύσις. Ἔχομεν : $\quad \frac{ημ2α}{1+συν2α} = \frac{2ημασυνα}{1+2συν^2α-1} = \frac{2ημασυνα}{2συν^2α} = εφα.$

Πρέπει : $\quad συν2α + 1 \neq 0$ ἢ $2συν^2α \neq 0$ ἢ $α \neq kπ + \frac{π}{2}, \quad k \in \mathbf{Z}.$

2. $$\frac{\eta\mu 2\alpha}{1-\sigma\upsilon\nu 2\alpha}=\sigma\varphi\alpha.$$

Λύσις. $$\frac{\eta\mu 2\alpha}{1-\sigma\upsilon\nu 2\alpha}=\frac{2\eta\mu\alpha\sigma\upsilon\nu\alpha}{1-1+2\eta\mu^2\alpha}=\frac{2\eta\mu\alpha\sigma\upsilon\nu\alpha}{2\eta\mu^2\alpha}=\sigma\varphi\alpha.$$

Πρέπει: $1-\sigma\upsilon\nu 2\alpha\neq 0$ ἢ $\eta\mu^2\alpha\neq 0$ ἢ $\alpha\neq k\pi$, $k\in\mathbf{Z}$.

3. $$\sigma\upsilon\nu^4\alpha-\eta\mu^4\alpha\equiv\sigma\upsilon\nu 2\alpha.$$

Λύσις. $\sigma\upsilon\nu^4\alpha-\eta\mu^4\alpha\equiv(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha)\equiv\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha\equiv\sigma\upsilon\nu 2\alpha.$

4. $$\sigma\varphi\alpha-\varepsilon\varphi\alpha=2\sigma\varphi 2\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\varphi\alpha-\varepsilon\varphi\alpha=\sigma\varphi\alpha-\frac{1}{\sigma\varphi\alpha}=\frac{\sigma\varphi^2\alpha-1}{\sigma\varphi\alpha}=2\cdot\frac{\sigma\varphi^2\alpha-1}{2\sigma\varphi\alpha}=2\cdot\sigma\varphi 2\alpha.$$

Πρέπει: $\alpha\neq k\pi$ καὶ $\alpha\neq k_1\pi+\frac{\pi}{2}$, $(k,k_1)\in\mathbf{Z}$.

5. $$\frac{\sigma\varphi\alpha-\varepsilon\varphi\alpha}{\sigma\varphi\alpha+\varepsilon\varphi\alpha}=\sigma\upsilon\nu 2\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\pi$ καὶ $\alpha\neq k_1\pi+\frac{\pi}{2}$, $(k,k_1)\in\mathbf{Z}$

$$\frac{\sigma\varphi\alpha-\varepsilon\varphi\alpha}{\sigma\varphi\alpha+\varepsilon\varphi\alpha}=\frac{\frac{\sigma\upsilon\nu\alpha}{\eta\mu\alpha}-\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha}}{\frac{\sigma\upsilon\nu\alpha}{\eta\mu\alpha}+\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha}}=\frac{\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha}{\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha}=\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha=\sigma\upsilon\nu 2\alpha.$$

6. $$\frac{1+\sigma\varphi^2\alpha}{2\sigma\varphi\alpha}=\sigma\tau\varepsilon\mu 2\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\pi$, καὶ $\alpha\neq k_1\frac{\pi}{2}$, $(k,k_1)\in\mathbf{Z}$.

$$\frac{1+\sigma\varphi^2\alpha}{2\sigma\varphi\alpha}=\frac{1+\frac{\sigma\upsilon\nu^2\alpha}{\eta\mu^2\alpha}}{\frac{2\sigma\upsilon\nu\alpha}{\eta\mu\alpha}}=\frac{\eta\mu^2\alpha+\sigma\upsilon\nu^2\alpha}{2\eta\mu\alpha\sigma\upsilon\nu\alpha}=\frac{1}{\eta\mu 2\alpha}=\sigma\tau\varepsilon\mu 2\alpha.$$

7. $$\frac{\sigma\varphi^2\alpha+1}{\sigma\varphi^2\alpha-1}=\tau\varepsilon\mu 2\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\left.\begin{matrix}\alpha\neq k\pi-\frac{\pi}{4}\\ \alpha\neq k_1\pi+\frac{\pi}{4}\end{matrix}\right\}\begin{matrix}, & k\in\mathbf{Z}\\ . & k_1\in\mathbf{Z}\end{matrix}$

$$\frac{\sigma\varphi^2\alpha+1}{\sigma\varphi^2\alpha-1}=\frac{\frac{\sigma\upsilon\nu^2\alpha}{\eta\mu^2\alpha}+1}{\frac{\sigma\upsilon\nu^2\alpha}{\eta\mu^2\alpha}-1}=\frac{\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha}{\sigma\upsilon\nu^2\alpha-\eta\mu^1\alpha}=\frac{1}{\sigma\upsilon\nu 2\alpha}=\tau\varepsilon\mu 2\alpha.$$

8. $$\textbf{εφ}(45^0-α)=\frac{συν2α}{1+ημ2α}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α\neq -45^0$ καὶ $\left.\begin{matrix} α\neq kπ+\dfrac{3π}{4} \\ α\neq(4k_1-1)\dfrac{π}{4}\end{matrix}\right\}\begin{matrix} k\in \mathbf{Z} \\ k_1\in\mathbf{Z}\end{matrix}$

$$εφ(45^0-α)=\frac{εφ45^0-εφα}{1+εφ45^0εφα}=\frac{1-εφα}{1+εφα}=\frac{1-\dfrac{ημα}{συνα}}{1+\dfrac{ημα}{συνα}}=\frac{συνα-ημα}{συνα+ημα}=$$

$$=\frac{(συνα-ημα)(συνα+ημα)}{(συνα+ημα)^2}=\frac{συν^2α-ημ^2α}{1+2ημασυνα}=\frac{συν2α}{1+ημ2α}.$$

9. $$\textbf{σφ}(45^0+α)=\frac{συν2α}{1+ημ2α}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α\neq -45^0$ καὶ $\left.\begin{matrix} α\neq kπ+\dfrac{3π}{4} \\ α\neq(4k_1-1)\dfrac{π}{4}\end{matrix}\right\}\begin{matrix} k\in \mathbf{Z} \\ k_1\in\mathbf{Z}\end{matrix}$

$$σφ(45^0+α)=\frac{σφ45^0σφα-1}{σφ45^0+σφα}=\frac{σφα-1}{σφα+1}=\frac{\dfrac{συνα}{ημα}-1}{\dfrac{συνα}{ημα}+1}=\frac{συνα-ημα}{συνα+ημα}=$$

$$=\frac{συν2α}{1+ημ2α}.$$

44. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ταυτότητες :

1. $$συν^2\left(\frac{π}{4}-α\right)-ημ^2\left(\frac{π}{4}-α\right)\equiv ημ2α.$$

Λύσις. Ἐὰν θέσωμεν $\frac{π}{4}-α=θ$, τὸ πρῶτον μέλος γίνεται :

$$συν^2\left(\frac{π}{4}-α\right)-ημ^2\left(\frac{π}{4}-α\right)\equiv συν^2θ-ημ^2θ\equiv συν2θ\equiv$$

$$\equiv συν2\left(\frac{π}{4}-α\right)\equiv συν\left(\frac{π}{2}-2α\right)\equiv ημ2α.$$

2. $$\textbf{εφ}(45^0+α)-εφ(45^0-α)=2εφ2ο.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α\neq\pm\frac{π}{4}$

$$εφ(45^0+α)-εφ(45^0-α)=\frac{εφ45^0+εφα}{1-εφ45^0εφα}-\frac{εφ45^0-εφα}{1+εφ45^0εφα}=$$

$$=\frac{1+εφα}{1-εφα}-\frac{1-εφα}{1+εφα}=\frac{(1+εφα)^2-(1-εφα)^2}{1-εφ^2α}=\frac{4εφα}{1-εφ^2α}=2\cdot εφ2α.$$

3. $$\textbf{εφ}(45^0+α)+\textbf{εφ}(45^0-α) = 2τεμ2α.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α \neq \pm \frac{π}{4}$:

$$εφ(45^0+α)+εφ(45^0-α) = \frac{1+εφα}{1-εφα} + \frac{1-εφα}{1+εφα} = \frac{(1+εφα)^2+(1-εφα)^2}{1-εφ^2α} =$$

$$= \frac{2+2εφ^2α}{1-εφ^2α} = 2 \cdot \frac{1+εφ^2α}{1-εφ^2α} = 2 \cdot \frac{1+\frac{ημ^2α}{συν^2α}}{1-\frac{ημ^2α}{συν^2α}} = 2 \cdot \frac{συν^2α+ημ^2α}{συν^2α-ημ^2α} =$$

$$= 2 \cdot \frac{1}{συν2α} = 2τεμ2α.$$

4. $$1-2ημ^2\left(\frac{π}{4}-\frac{α}{2}\right) \equiv ημα.$$

Λύσις. Θέτομεν $\frac{π}{4}-\frac{α}{2} = θ$ καὶ ἔχομεν:

$$1-2ημ^2\left(\frac{π}{4}-\frac{α}{2}\right) \equiv 1-2ημ^2θ \equiv συν2θ \equiv συν2\left(\frac{π}{4}-\frac{α}{2}\right) \equiv$$

$$\equiv συν\left(\frac{π}{2}-α\right) \equiv ημα.$$

5. $$\frac{1-εφ^2(45^0-α)}{1+εφ^2(45^0-α)} = ημ2α.$$

Λύσις. Ἂν $α \neq -\frac{π}{4}$, θὰ ἔχωμεν:

$$εφ^2(45^0-α) = \left(\frac{εφ45^0-εφα}{1+εφ45^0εφα}\right)^2 = \frac{(1-εφα)^2}{(1+εφα)^2} = \frac{(συνα-ημα)^2}{(συνα+ημα)^2} =$$

$$= \frac{συν^2α+ημ^2α-2ημασυνα}{συν^2α+ημ^2α+2ημασυνα} = \frac{1-ημ2α}{1+ημ2α},$$

καὶ κατ' ἀκολουθίαν θὰ ἔχωμεν διαδοχικῶς:

$$\frac{1-εφ^2(45^0-α)}{1+εφ^2(45^0-α)} = \frac{1-\frac{1-ημ2α}{1+ημ2α}}{1+\frac{1-ημ2α}{1+ημ2α}} = \frac{2ημ2α}{2} = ημ2α.$$

6. $$\frac{συνα+ημα}{συνα-ημα} - \frac{συνα-ημα}{συνα+ημα} = 2εφ2α.$$

Λύσις. Ἐὰν καλέσωμεν Κ τὸ α' μέλος θὰ ἔχωμεν:

$$Κ = \frac{(συνα+ημα)^2-(συνα-ημα)^2}{συν^2α-ημ^2α} = \frac{4συναημα}{συν2α} =$$

$$= 2 \cdot \frac{ημ2α}{συν2α} = 2εφ2α.$$

Πότε ἔχουν ἔννοιαν τὰ κλάσματα τοῦ α' μέλους τῆς (1);

7. $$\frac{\eta\mu\alpha+\eta\mu2\alpha}{1+\sigma\upsilon\nu\alpha+\sigma\upsilon\nu2\alpha}=\varepsilon\varphi\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{\eta\mu\alpha+\eta\mu2\alpha}{1+\sigma\upsilon\nu\alpha+\sigma\upsilon\nu2\alpha}=\frac{\eta\mu\alpha+2\eta\mu\alpha\sigma\upsilon\nu\alpha}{1+\sigma\upsilon\nu\alpha+2\sigma\upsilon\nu^2\alpha-1}=\frac{\eta\mu\alpha(1+2\sigma\upsilon\nu\alpha)}{\sigma\upsilon\nu\alpha(1+2\sigma\upsilon\nu\alpha)}=\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha}=\varepsilon\varphi\alpha.$$

ἂν $$\alpha\neq k\pi+\frac{\pi}{2} \quad \text{καὶ} \quad \alpha\neq 2k_1\pi\pm\frac{2\pi}{3}, \qquad (k,\ k_1)\in \mathbf{Z}.$$

8. $$\frac{1-\sigma\upsilon\nu2\alpha+\eta\mu2\alpha}{1+\sigma\upsilon\nu2\alpha+\eta\mu2\alpha}=\varepsilon\varphi\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{1-\sigma\upsilon\nu2\alpha+\eta\mu2\alpha}{1+\sigma\upsilon\nu2\alpha+\eta\mu2\alpha}=\frac{1-1+2\eta\mu^2\alpha+2\eta\mu\alpha\sigma\upsilon\nu\alpha}{1+2\sigma\upsilon\nu^2\alpha-1+2\eta\mu\alpha\sigma\upsilon\nu\alpha}=$$

$$=\frac{2\eta\mu\alpha(\eta\mu\alpha+\sigma\upsilon\nu\alpha)}{2\sigma\upsilon\nu\alpha(\sigma\upsilon\nu\alpha+\eta\mu\alpha)}=\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha}=\varepsilon\varphi\alpha.$$

ἂν $$\alpha\neq k\pi+\frac{\pi}{2} \quad \text{καὶ} \quad \alpha\neq-\frac{\pi}{4}+k_1\pi, \qquad (k,\ k_1)\in \mathbf{Z}.$$

9. $$\varepsilon\varphi(\alpha+30^0)\varepsilon\varphi(\alpha-30^0)=\frac{1-2\sigma\upsilon\nu2\alpha}{1+2\sigma\upsilon\nu2\alpha}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq 60^0$ καὶ $\alpha\neq 120$:

$$\varepsilon\varphi(\alpha+30^0)\varepsilon\varphi(\alpha-30^0)=\frac{\varepsilon\varphi\alpha+\varepsilon\varphi30^2}{1-\varepsilon\varphi\alpha\varepsilon\varphi30^0}\cdot\frac{\varepsilon\varphi\alpha-\varepsilon\varphi30^0}{1+\varepsilon\varphi\alpha\varepsilon\varphi30^0}=$$

$$=\frac{\varepsilon\varphi\alpha+\frac{\sqrt{3}}{3}}{1-\frac{\sqrt{3}}{3}\varepsilon\varphi\alpha}\cdot\frac{\varepsilon\varphi\alpha-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}\varepsilon\varphi\alpha}=\frac{\varepsilon\varphi^2\alpha-\frac{1}{3}}{1-\frac{1}{3}\varepsilon\varphi^2\alpha}=\frac{3\varepsilon\varphi^2\alpha-1}{3-\varepsilon\varphi^2\alpha}=$$

$$=\frac{3\cdot\frac{\eta\mu^2\alpha}{\sigma\upsilon\nu^2\alpha}-1}{3-\frac{\eta\mu^2\alpha}{\sigma\upsilon\nu^2\alpha}}=\frac{3\eta\mu^2\alpha-\sigma\upsilon\nu^2\alpha}{3\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha}=\frac{3(1-\sigma\upsilon\nu^2\alpha)-\sigma\upsilon\nu^2\alpha}{2\sigma\upsilon\nu^2\alpha-1+\sigma\upsilon\nu^2\alpha}=$$

$$=\frac{3-4\sigma\upsilon\nu^2\alpha}{4\sigma\upsilon\nu^2\alpha-1}=\frac{1-4\sigma\upsilon\nu^2\alpha+2}{1+4\sigma\upsilon\nu^2\alpha-2}=\frac{1-2(2\sigma\upsilon\nu^2\alpha-1)}{1+2(2\sigma\upsilon\nu^2\alpha-1)}=\frac{1-2\sigma\upsilon\nu2\alpha}{1+2\sigma\upsilon\nu2\alpha}.$$

45. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ἰσότητες :

1. $$\frac{\eta\mu3\alpha}{\eta\mu\alpha}-\frac{\sigma\upsilon\nu3\alpha}{\sigma\upsilon\nu\alpha}=2.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\pi$ καὶ $\alpha\neq k_1\pi+\frac{\pi}{2}$, $(k,k_1)\in\mathbf{Z}$

$$\frac{\eta\mu3\alpha}{\eta\mu\alpha}-\frac{\sigma\upsilon\nu3\alpha}{\sigma\upsilon\nu\alpha}=\frac{3\eta\mu\alpha-4\eta\mu^3\alpha}{\eta\mu\alpha}-\frac{4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\alpha}=$$

$$= 3-4\eta\mu^2\alpha-(4\sigma\upsilon\nu^2\alpha-3) = 3-4\eta\mu^2\alpha-4\sigma\upsilon\nu^2\alpha+3 = 6-4(\eta\mu^2\alpha+\sigma\upsilon\nu^2\alpha) =$$
$$= 6-4\cdot 1 = 6-4 = 2.$$

2. $$\frac{\mathbf{3\sigma\upsilon\nu\alpha+\sigma\upsilon\nu 3\alpha}}{\mathbf{3\eta\mu\alpha-\eta\mu 3\alpha}} = \mathbf{\sigma\varphi^3\alpha.}$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq k\pi$, $k \in \mathbf{Z}$

$$\frac{3\sigma\upsilon\nu\alpha+\sigma\upsilon\nu 3\alpha}{3\eta\mu\alpha-\eta\mu 3\alpha} = \frac{3\sigma\upsilon\nu\alpha+4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha}{3\eta\mu\alpha-(3\eta\mu\alpha-4\eta\mu^3\alpha)} = \frac{4\sigma\upsilon\nu^3\alpha}{4\eta\mu^3\alpha} = \sigma\varphi^3\alpha.$$

3. $$\frac{\mathbf{\eta\mu 3\alpha+\eta\mu^3\alpha}}{\mathbf{\sigma\upsilon\nu^3\alpha-\sigma\upsilon\nu 3\alpha}} = \mathbf{\sigma\varphi\alpha.}$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq k\pi+\frac{\pi}{2}$, $\alpha \neq k_1\pi$, $(k,k_1)\in\mathbf{Z}$

$$\frac{\eta\mu 3\alpha+\eta\mu^3\alpha}{\sigma\upsilon\nu^3\alpha-\sigma\upsilon\nu 3\alpha} = \frac{3\eta\mu\alpha-4\eta\mu^3\alpha+\eta\mu^3\alpha}{\sigma\upsilon\nu^3\alpha-(4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha)} = \frac{3\eta\mu\alpha(1-\eta\mu^2\alpha)}{3\sigma\upsilon\nu\alpha(1-\sigma\upsilon\nu^2\alpha)} =$$
$$= \frac{\eta\mu\alpha\cdot\sigma\upsilon\nu^2\alpha}{\sigma\upsilon\nu\alpha\cdot\eta\mu^2\alpha} = \frac{\sigma\upsilon\nu\alpha}{\eta\mu\alpha} = \sigma\varphi\alpha.$$

4. $$\frac{\mathbf{\sigma\upsilon\nu^3\alpha-\sigma\upsilon\nu 3\alpha}}{\mathbf{\sigma\upsilon\nu\alpha}} + \frac{\mathbf{\eta\mu^3\alpha+\eta\mu 3\alpha}}{\mathbf{\sigma\upsilon\nu\alpha}} = \mathbf{3.}$$

Λύσις. Ἔχομεν διαδοχικῶς, $\alpha \neq k\pi$ καὶ $\alpha \neq k_1\pi+\frac{\pi}{2}$, $(k, k_1)\in\mathbf{Z}$

$$+\ \frac{\sigma\upsilon\nu^3\alpha-\sigma\upsilon\nu 3\alpha}{\sigma\upsilon\nu\alpha} + \frac{\eta\mu^3\alpha+\eta\mu 3\alpha}{\eta\mu\alpha} = \frac{\sigma\upsilon\nu^3\alpha-4\sigma\upsilon\nu^3\alpha+3\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\alpha} +$$
$$+\ \frac{\eta\mu^3\alpha+3\eta\mu\alpha-4\eta\mu^3\alpha}{\eta\mu\alpha} = \frac{3\sigma\upsilon\nu\alpha(1-\sigma\upsilon\nu^2\alpha)}{\sigma\upsilon\nu\alpha} + \frac{3\eta\mu\alpha(1-\eta\mu^2\alpha)}{\eta\mu\alpha} =$$
$$= 3\eta\mu^2\alpha+3\sigma\upsilon\nu^2\alpha = 3.$$

5. $$\mathbf{4\eta\mu^3\alpha\sigma\upsilon\nu 3\alpha+4\sigma\upsilon\nu^3\alpha\eta\mu 3\alpha \equiv 3\eta\mu 4\alpha.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$4\eta\mu^3\alpha\sigma\upsilon\nu 3\alpha+4\sigma\upsilon\nu^3\alpha\eta\mu 3\alpha \equiv 4\eta\mu^3\alpha(4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha)+4\sigma\upsilon\nu^3\alpha(3\eta\mu\alpha-4\eta\mu^3\alpha) \equiv$$
$$\equiv 16\eta\mu^3\alpha\sigma\upsilon\nu^3\alpha-12\eta\mu^3\alpha\sigma\upsilon\nu\alpha+12\sigma\upsilon\nu^3\alpha\eta\mu\alpha-16\eta\mu^3\alpha\sigma\upsilon\nu^3\alpha \equiv$$
$$\equiv 12\sigma\upsilon\nu^3\alpha\eta\mu\alpha-12\eta\mu^3\alpha\sigma\upsilon\nu\alpha \equiv 12\eta\mu\alpha\sigma\upsilon\nu\alpha(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha) \equiv$$
$$\equiv 12\eta\mu\alpha\sigma\upsilon\nu\alpha\cdot\sigma\upsilon\nu 2\alpha \equiv 6\cdot 2\eta\mu\alpha\sigma\upsilon\nu\alpha\cdot\sigma\upsilon\nu 2\alpha \equiv 6\cdot\eta\mu 2\alpha\sigma\upsilon\nu 2\alpha \equiv$$
$$\equiv 3\cdot 2\eta\mu 2\alpha\sigma\upsilon\nu 2\alpha \equiv 3\cdot\eta\mu 2(2\alpha) \equiv 3\eta\mu 4\alpha.$$

6. $$\mathbf{\sigma\upsilon\nu^3\alpha\sigma\upsilon\nu 3\alpha+\eta\mu^3\alpha\eta\mu 3\alpha \equiv \sigma\upsilon\nu^3 2\alpha.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu^3\alpha\sigma\upsilon\nu 3\alpha+\eta\mu^3\alpha\eta\mu 3\alpha \equiv \sigma\upsilon\nu^3\alpha(4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha)+\eta\mu^3\alpha(3\eta\mu\alpha-4\eta\mu^3\alpha) \equiv$$
$$\equiv 4\sigma\upsilon\nu^6\alpha-3\sigma\upsilon\nu^4\alpha+3\eta\mu^4\alpha-4\eta\mu^6\alpha \equiv 4(\sigma\upsilon\nu^6\alpha-\eta\mu^6\alpha)-3(\sigma\upsilon\nu^4\alpha-\eta\mu^4\alpha) \equiv$$
$$\equiv 4(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha)(\sigma\upsilon\nu^4\alpha+\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha+\eta\mu^4\alpha)-3(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha) \equiv$$

$$\equiv 4\sigma\upsilon\nu 2\alpha[(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha)^2+3\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha]-3\sigma\upsilon\nu 2\alpha \equiv$$
$$\equiv \sigma\upsilon\nu 2\alpha[4\sigma\upsilon\nu^2 2\alpha+12\sigma\upsilon\nu^2\alpha(1-\sigma\upsilon\nu^2\alpha)-3] \equiv$$
$$\equiv \sigma\upsilon\nu 2\alpha(4\sigma\upsilon\nu^2 2\alpha+12\sigma\upsilon\nu^2\alpha-12\sigma\upsilon\nu^4\alpha-3) \equiv$$
$$\equiv \sigma\upsilon\nu 2\alpha(4\sigma\upsilon\nu^2 2\alpha+12\sigma\upsilon\nu^2\alpha-12\sigma\upsilon\nu^4\alpha-3) \equiv$$
$$\equiv \sigma\upsilon\nu 2\alpha[4(2\sigma\upsilon\nu^2\alpha-1)^2+12\sigma\upsilon\nu^2\alpha-12\sigma\upsilon\nu^4\alpha-3] \equiv$$
$$\equiv \sigma\upsilon\nu 2\alpha(16\sigma\upsilon\nu^4\alpha-16\sigma\upsilon\nu^2\alpha+4+12\sigma\upsilon\nu^2\alpha-12\sigma\upsilon\nu^4\alpha-3) \equiv$$
$$\equiv \sigma\upsilon\nu 2\alpha(4\sigma\upsilon\nu^4\alpha-4\sigma\upsilon\nu^2\alpha+1) \equiv \sigma\upsilon\nu 2\alpha \cdot \sigma\upsilon\nu^2 2\alpha \equiv \sigma\upsilon\nu^3 2\alpha.$$

7. **$4\eta\mu\alpha\eta\mu(60^0+\alpha)\eta\mu(60^0-\alpha) \equiv \eta\mu 3\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$4\eta\mu\alpha\eta\mu(60^0+\alpha)\eta\mu(60^0-\alpha) \equiv 4\eta\mu\alpha[\eta\mu^2 60-\eta\mu^2\alpha] \equiv$$
$$\equiv 4\eta\mu\alpha\left(\frac{3}{4}-\eta\mu^2\alpha\right) \equiv 3\eta\mu\alpha-4\eta\mu^3\alpha \equiv \eta\mu 3\alpha.$$

8. **$4\sigma\upsilon\nu\alpha\sigma\upsilon\nu(60^0+\alpha)\sigma\upsilon\nu(60^0-\alpha) \equiv \sigma\upsilon\nu 3\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$4\sigma\upsilon\nu\alpha\sigma\upsilon\nu(60^0+\alpha)\sigma\upsilon\nu(60^0-\alpha) \equiv 4\sigma\upsilon\nu\alpha[\sigma\upsilon\nu^2\alpha-\eta\mu^2 60] \equiv$$
$$\equiv 4\sigma\upsilon\nu\alpha\left(\sigma\upsilon\nu^2\alpha-\frac{3}{4}\right) \equiv 4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha \equiv \sigma\upsilon\nu 3\alpha.$$

9. **$\varepsilon\varphi\alpha\varepsilon\varphi(60^0+\alpha)\varepsilon\varphi(60^0-\alpha) = \varepsilon\varphi 3\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq 30^0$ καὶ $\alpha \neq -30^0$.

$$\varepsilon\varphi\alpha\varepsilon\varphi(60^0+\alpha)\varepsilon\varphi(60^0-\alpha) = \varepsilon\varphi\alpha \cdot \frac{\varepsilon\varphi 60^0+\varepsilon\varphi\alpha}{1-\varepsilon\varphi 60^0\varepsilon\varphi\alpha} \cdot \frac{\varepsilon\varphi 60^0-\varepsilon\varphi\alpha}{1+\varepsilon\varphi 60^0\varepsilon\varphi\alpha} =$$
$$= \varepsilon\varphi\alpha \cdot \frac{\varepsilon\varphi^2 60^0-\varepsilon\varphi^2\alpha}{1-\varepsilon\varphi^2 60^0\varepsilon\varphi^2\alpha} = \varepsilon\varphi\alpha \cdot \frac{3-\varepsilon\varphi^2\alpha}{1-3\varepsilon\varphi^2\alpha} = \frac{3\varepsilon\varphi\alpha-\varepsilon\varphi^3\alpha}{1-3\varepsilon\varphi^2\alpha} = \varepsilon\varphi 3\alpha.$$

10. **$\sigma\varphi\alpha+\sigma\varphi(60^0+\alpha)-\sigma\varphi(60^0-\alpha) \equiv 3\sigma\varphi 3\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq \pm 60^0$

$$\sigma\varphi\alpha+\sigma\varphi(60^0+\alpha)-\sigma\varphi(60^0-\alpha) = \sigma\varphi\alpha+\frac{\sigma\varphi 60^0\sigma\varphi\alpha-1}{\sigma\varphi 60^0+\sigma\varphi\alpha}-\frac{\sigma\varphi 60^0\sigma\varphi\alpha+1}{\sigma\varphi\alpha-\sigma\varphi 60^0} =$$
$$= \sigma\varphi\alpha+\frac{\frac{\sqrt{3}}{3}\sigma\varphi\alpha-1}{\frac{\sqrt{3}}{3}+\sigma\varphi\alpha}-\frac{\frac{\sqrt{3}}{3}\sigma\varphi\alpha+1}{\sigma\varphi\alpha-\frac{\sqrt{3}}{3}} = \sigma\varphi\alpha+\frac{\sqrt{3}\,\sigma\varphi\alpha-3}{3\sigma\varphi\alpha+\sqrt{3}}-$$
$$-\frac{\sqrt{3}\,\sigma\varphi\alpha+3}{3\sigma\varphi\alpha-\sqrt{3}} =$$
$$= \sigma\varphi\alpha+\frac{3\sqrt{3}\sigma\varphi^2\alpha-9\sigma\varphi\alpha-3\sigma\varphi\alpha+3\sqrt{3}-3\sqrt{3}\,\sigma\varphi^2\alpha-9\sigma\varphi\alpha-3\sigma\varphi\alpha-3\sqrt{3}}{9\sigma\varphi^2\alpha-2} =$$
$$= \sigma\varphi\alpha+\frac{-24\sigma\varphi\alpha}{9\sigma\varphi^2\alpha-3} = \sigma\varphi\alpha-\frac{8\sigma\varphi\alpha}{3\sigma\varphi^2\alpha-1} = \frac{3\sigma\varphi^3\alpha-\sigma\varphi\alpha-8\sigma\varphi\alpha}{3\sigma\varphi^2\alpha-1} =$$
$$= \frac{3\sigma\varphi^3\alpha-9\sigma\varphi\alpha}{3\sigma\varphi^2\alpha-1} = 3 \cdot \frac{\sigma\varphi^3\alpha-3\,\sigma\varphi\alpha}{3\sigma\varphi^2\alpha-1} = 3 \cdot \sigma\varphi 3\alpha.$$

11. $$εφ3α - εφ2α - εφα = εφ3α εφ2α εφα.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α \neq \frac{π}{6}$, $α \neq \frac{π}{4}$, $α \neq \frac{π}{2}$

$$εφ3α = εφ(2α+α) = \frac{εφ2α + εφα}{1 - εφ2α εφα} \quad ἤ$$

$$εφ3α(1 - εφ2α εφα) = εφ2α + εφα \quad ἤ \quad εφ3α - εφ3α εφ2α εφα = εφ2α + εφα,$$

ἐξ οὗ: $$εφ3α - εφ2α - εφα = εφ3α εφ2α εφα.$$

46. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ἰσότητες:

1. $$\frac{εφ^2 2x}{2 + εφ^2 2x} = \frac{2εφ^2 x}{1 + εφ^3 x}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $x \neq kπ + \frac{π}{4}$, $x \neq k_1 π + \frac{π}{2}$, $(k, k_1) \in \mathbf{Z}$.

$$\frac{εφ^2 2x}{2 + εφ^2 2x} = \frac{\left(\frac{2εφx}{1 - εφ^2 x}\right)^2}{2 + \left(\frac{2εφx}{1 - εφ^2 x}\right)^2} = \frac{\frac{4εφ^2 x}{(1 - εφ^2 x)^2}}{\frac{2(1 - εφ^2 x) + 4εφ^2 x}{(1 - εφ^2 x)^2}} =$$

$$= \frac{4εφ^2 x}{2 - 4εφ^2 x + 2εφ^4 x + 4εφ^2 x} = \frac{4εφ^2 x}{2 + 2εφ^4 x} = \frac{2εφ^2 x}{1 + εφ^2 x}.$$

2. $$εφ^2 x + σφ^2 x \equiv \frac{2(3 + συν4x)}{1 - συν4x}.$$

Λύσις. Ἐὰν $x \neq π + kπ$ καὶ $x \neq \frac{π}{2} + k_1 π$, θὰ ἔχωμεν:

$$εφ^2 x + σφ^2 x = \frac{ημ^2 x}{συν^2 x} + \frac{συν^2 x}{ημ^2 x} = \frac{ημ^4 x + συν^4 x}{ημ^2 x συν^2 x} =$$

$$= \frac{(ημ^2 x + συν^2 x)^2 - 2ημ^2 x συν^2 x}{ημ^2 x συν^2 x} = \frac{1 - 2ημ^2 x συν^2 x}{ημ^2 x συν^2 x} =$$

$$= \frac{1 - \frac{1}{2} ημ^2 2x}{\frac{1}{4} ημ^2 2x} = \frac{1 - \frac{1}{2}\left(\frac{1 - συν4x}{2}\right)}{\frac{1}{4}\left(\frac{1 - συν4x}{2}\right)} = \frac{1 - \frac{1 - συν4x}{4}}{\frac{1 - συν4x}{8}} =$$

$$= \frac{2(3 + συν4x)}{1 - συν4x}.$$

3. $$\frac{1}{εφ3α - εφα} - \frac{1}{σφ3α - σφα} = σφ2α.$$

Λύσις. Θὰ ἔχωμεν διαδοχικῶς, ἂν $α \neq k\frac{π}{2}$, $k \in \mathbf{Z}$.

$$\frac{1}{\varepsilon\varphi 3\alpha-\varepsilon\varphi\alpha}-\frac{1}{\sigma\varphi 3\alpha-\sigma\varphi\alpha}=\frac{1}{\frac{\eta\mu 3\alpha}{\sigma\upsilon\nu 3\alpha}-\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha}}-\frac{1}{\frac{\sigma\upsilon\nu 3\alpha}{\eta\mu 3\alpha}-\frac{\sigma\upsilon\nu\alpha}{\eta\mu\alpha}}=$$

$$=\frac{\sigma\upsilon\nu\alpha\sigma\upsilon\nu 3\alpha}{\eta\mu 3\alpha\sigma\upsilon\nu\alpha-\eta\mu\alpha\sigma\upsilon\nu 3\alpha}-\frac{\eta\mu\alpha\eta\mu 3\alpha}{\eta\mu\alpha\sigma\upsilon\nu 3\alpha-\sigma\upsilon\nu\alpha\eta\mu 3\alpha}=$$

$$=\frac{\sigma\upsilon\nu\alpha\sigma\upsilon\nu 3\alpha}{\eta\mu(3\alpha-\alpha)}-\frac{\eta\mu\alpha\eta\mu 3\alpha}{\eta\mu(\alpha-3\alpha)}=\frac{\sigma\upsilon\nu\alpha\sigma\upsilon\nu 3\alpha}{\eta\mu 2\alpha}+\frac{\eta\mu\alpha\eta\mu 3\alpha}{\eta\mu 2\alpha}=$$

$$=\frac{\sigma\upsilon\nu\alpha\sigma\upsilon\nu 3\alpha+\eta\mu\alpha\eta\mu 3\alpha}{\eta\mu 2\alpha}=\frac{\sigma\upsilon\nu(3\alpha-\alpha)}{\sigma\upsilon\nu(3\alpha-\alpha)}=\frac{\sigma\upsilon\nu 2\alpha}{\eta\mu 2\alpha}=\sigma\varphi 2\alpha.$$

4. $$\boldsymbol{\frac{\sigma\varphi\alpha}{\sigma\varphi\alpha-\sigma\varphi 3\alpha}+\frac{\varepsilon\varphi\alpha}{\varepsilon\varphi\alpha-\varepsilon\varphi 3\alpha}=1.}$$

Λύσις. Θὰ ἔχωμεν διαδοχικῶς, ἂν $\alpha\neq k\frac{\pi}{2}$, $k\in\mathbf{Z}$.

$$\frac{\sigma\varphi\alpha}{\sigma\varphi\alpha-\sigma\varphi 3\alpha}+\frac{\varepsilon\varphi\alpha}{\varepsilon\varphi\alpha-\varepsilon\varphi 3\alpha}=\frac{\frac{1}{\varepsilon\varphi\alpha}}{\frac{1}{\varepsilon\varphi\alpha}-\frac{1}{\varepsilon\varphi 3\alpha}}+\frac{\varepsilon\varphi\alpha}{\varepsilon\varphi\alpha-\varepsilon\varphi 3\alpha}=$$

$$=\frac{\varepsilon\varphi 3\alpha}{\varepsilon\varphi 3\alpha-\varepsilon\varphi\alpha}+\frac{\varepsilon\varphi\alpha}{\varepsilon\varphi\alpha-\varepsilon\varphi 3\alpha}=\frac{\varepsilon\varphi 3\alpha-\varepsilon\varphi\alpha}{\varepsilon\varphi 3\alpha-\varepsilon\varphi\alpha}=1.$$

5. $$\boldsymbol{\frac{1}{\varepsilon\varphi 3\alpha+\varepsilon\varphi\alpha}-\frac{1}{\sigma\varphi 3\alpha+\sigma\varphi\alpha}=\sigma\varphi 4\alpha.}$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\frac{\pi}{4}$, $k\in\mathbf{Z}$.

$$\frac{1}{\varepsilon\varphi 3\alpha+\varepsilon\varphi\alpha}-\frac{1}{\sigma\varphi 3\alpha+\sigma\varphi\alpha}=\frac{1}{\varepsilon\varphi 3\alpha+\varepsilon\varphi\alpha}-\frac{\varepsilon\varphi 3\alpha\varepsilon\varphi\alpha}{\varepsilon\varphi 3\alpha+\varepsilon\varphi\alpha}=$$

$$=\frac{1-\varepsilon\varphi 3\alpha\varepsilon\varphi\alpha}{\varepsilon\varphi 3\alpha+\varepsilon\varphi\alpha}=\frac{1}{\varepsilon\varphi 4\alpha}=\sigma\varphi 4\alpha.$$

6. $$\boldsymbol{4(\sigma\upsilon\nu^6\alpha+\eta\mu^6\alpha)\equiv 1+3\sigma\upsilon\nu^2 2\alpha.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$4(\sigma\upsilon\nu^6\alpha+\eta\mu^6\alpha)\equiv 4[(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)(\sigma\upsilon\nu^4\alpha-\sigma\upsilon\nu^2\alpha\eta\mu^2\alpha+\eta\mu^4\alpha)\equiv$$

$$\equiv 4(\sigma\upsilon\nu^4\alpha+\eta\mu^4\alpha-\sigma\upsilon\nu^2\alpha\eta\mu^2\alpha)\equiv 4[(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)^2-3\sigma\upsilon\nu^2\alpha\eta\mu^2\alpha]\equiv$$

$$\equiv 4(1-3\sigma\upsilon\nu^2\eta\mu^2\alpha)\equiv 4[1-3\sigma\upsilon\nu^2\alpha(1-\sigma\upsilon\nu^2\alpha)]\equiv 4(1-3\sigma\upsilon\nu^2\alpha+3\sigma\upsilon\nu^4\alpha)\equiv$$

$$\equiv 12\sigma\upsilon\nu^4\alpha-12\sigma\upsilon\nu^2\alpha+4=1+12\sigma\upsilon\nu^4\alpha-12\sigma\upsilon\nu^2\alpha+3\equiv$$

$$\equiv 1+3(4\sigma\upsilon\nu^4\alpha-4\sigma\upsilon\nu^2\alpha+1)\equiv 1+3\sigma\upsilon\nu^2 2\alpha.$$

47. Νὰ ἀποδειχθῇ ὅτι :

1. $$\boldsymbol{\eta\mu^2 72^0-\eta\mu^2 60^0=}\frac{\sqrt{5}-1}{8}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$ημ^2 72^0 - ημ^2 60^0 = \frac{\left(\sqrt{10+2\sqrt{5}}\right)^2}{16} - \frac{3}{4} = \frac{10+2\sqrt{5}}{13} - \frac{12}{16} = \frac{2\sqrt{5}-2}{16} = \frac{\sqrt{5}-1}{8}.$$

2. $$\mathbf{ημ\,\frac{\pi}{10} + ημ\,\frac{13\pi}{10} = -\frac{1}{4}.}$$

Λύσις : Ἔχομεν διαδοχικῶς :

$$ημ\,\frac{\pi}{10} + ημ\,\frac{13\pi}{10} = ημ18^0 + ημ234^0 = ημ18^0 + ημ(180^0 + 54^0) =$$

$$= ημ18^0 - ημ54^0 = \frac{\sqrt{5}-1}{4} - \frac{\sqrt{5}+1}{4} = \frac{-2}{4} = -\frac{1}{2}.$$

3. $$\mathbf{ημ\,\frac{\pi}{10}\,ημ\,\frac{13\pi}{10} = -\frac{1}{4}.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$ημ\,\frac{\pi}{10} \cdot ημ\,\frac{13\pi}{10} = \frac{\sqrt{5}-1}{4}\left(-\frac{\sqrt{5}+1}{4}\right) = -\frac{5-1}{16} = -\frac{4}{16} = -\frac{1}{4}$$

4. $$\mathbf{ημ\,\frac{\pi}{5} \cdot ημ\,\frac{2\pi}{5} \cdot ημ\,\frac{3\pi}{5} \cdot ημ\,\frac{4\pi}{5} = \frac{5}{16}.}$$

Λύσις : Ἔχομεν διαδοχικῶς : $ημ\,\frac{4\pi}{5} = ημ\,\frac{\pi}{5}$, $ημ\,\frac{3\pi}{5} = ημ\,\frac{2\pi}{5}$.

Ἄρα : $$ημ\,\frac{\pi}{5} \cdot ημ\,\frac{2\pi}{5} \cdot ημ\,\frac{3\pi}{5} \cdot ημ\,\frac{4\pi}{5} = ημ^2\frac{\pi}{5} \cdot ημ^2\,\frac{2\pi}{5} = ημ^2 \cdot 36^0 \cdot ημ^2 72^0 =$$

$$= \left(\frac{1}{4}\sqrt{10-2\sqrt{5}}\right)^2 \cdot \left(\frac{1}{4}\sqrt{10+2\sqrt{5}}\right)^2 =$$

$$= \frac{10-2\sqrt{5}}{16} \cdot \frac{10+2\sqrt{5}}{16} = \frac{100-20}{256} = \frac{80}{256} = \frac{10}{32} = \frac{5}{16}.$$

48. Ἐὰν $\alpha = 18^0$, νὰ ἀποδειχθῇ ὅτι :

1. $$\mathbf{συν2\alpha + 2συν4\alpha + 3συν6\alpha + 4συν8\alpha = -\frac{5}{2}.}$$

Λύσις. Θὰ ἔχωμεν :

$$συν2\alpha = συν(2 \cdot 18^0) = συν35^0 = \frac{\sqrt{5}+1}{4}$$

$$συν4\alpha = συν(4 \cdot 18^0) = συν72^0 = \frac{\sqrt{5}-1}{4}$$

$$συν6\alpha = συν(6 \cdot 18^0) = συν108^0 = -συν72^0 = -\frac{\sqrt{5}-1}{4}$$

$$συν8\alpha = συν(8 \cdot 18^0) = συν144^0 = -συν36^0 = -\frac{\sqrt{5}+1}{4}.$$

Κατ' ἀκολουθίαν:

$$\sigma\upsilon\nu 2\alpha + 2\sigma\upsilon\nu 4\alpha + 3\sigma\upsilon\nu 6\alpha + 4\sigma\upsilon\nu 8\alpha =$$

$$= \frac{\sqrt{5}+1}{4} + 2\cdot\frac{\sqrt{5}-1}{4} + 3\cdot\left(-\frac{\sqrt{5}-1}{4}\right) + 4\left(-\frac{\sqrt{5}+1}{4}\right) =$$

$$= \frac{\sqrt{5}+1+2\sqrt{5}-2-3\sqrt{5}+3-4\sqrt{5}-4}{4} = \frac{-4\sqrt{5}-2}{4} = -\frac{4\sqrt{5}+2}{4}.$$

2. $$\eta\mu^2\alpha + 2\eta\mu^2 2\alpha + 3\eta\mu^2 3\alpha + 4\eta\mu^2 4\alpha = \frac{21+2\sqrt{5}}{4} = \frac{11+\sqrt{5}}{2}.$$

Λύσις. Θὰ ἔχωμεν:

$$\eta\mu^2\alpha = \eta\mu^2 18^0 = \frac{(\sqrt{5}-1)^2}{16} = \frac{6-2\sqrt{5}}{16},$$

$$\eta\mu^2 2\alpha = \eta\mu^2 36^0 = \frac{\left(\sqrt{10-2\sqrt{5}}\right)^2}{16} = \frac{10-2\sqrt{5}}{16},$$

$$\eta\mu^2 3\alpha = \eta\mu^2 54^0 = \frac{(\sqrt{5}+1)^2}{16} = \frac{6+2\sqrt{5}}{16},$$

$$\eta\mu^2 4\alpha = \eta\mu^2 72^2 = \frac{\left(\sqrt{10+2\sqrt{5}}\right)^2}{16} = \frac{10+2\sqrt{5}}{16}.$$

Κατ' ἀκολουθίαν: $\eta\mu^2\alpha + 2\eta\mu^2 2\alpha + 3\eta\mu^2 3\alpha + 4\eta\mu^2 4\alpha =$

$$= \frac{6-2\sqrt{5}}{16} + 2\cdot\frac{10-2\sqrt{5}}{16} + 3\cdot\frac{6+2\sqrt{5}}{16} + 4\cdot\frac{10+2\sqrt{5}}{16} =$$

$$= \frac{6-2\sqrt{5}+20-4\sqrt{5}+18+6\sqrt{5}+40+8\sqrt{5}}{16} = \frac{8\sqrt{5}+84}{16} = \frac{21+2\sqrt{5}}{4}.$$

3. $$\sigma\upsilon\nu\alpha\cdot\sigma\upsilon\nu 2\alpha\cdot\sigma\upsilon\nu 3\alpha\cdot\sigma\upsilon\nu 4\alpha = \frac{\sqrt{5}}{16}.$$

Λύσις. Θὰ ἔχωμεν διαδοχικῶς:

$$\sigma\upsilon\nu\alpha\cdot\sigma\upsilon\nu 2\alpha\cdot\sigma\upsilon\nu 3\alpha\cdot\sigma\upsilon\nu 4\alpha = \frac{\sqrt{10+2\sqrt{5}}}{4}\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{10-2\sqrt{5}}}{4}\cdot\frac{\sqrt{5}-1}{4} =$$

$$= \frac{5-1}{16}\cdot\frac{\sqrt{100-20}}{16} = \frac{4}{16}\cdot\frac{\sqrt{80}}{16} = \frac{1}{4}\cdot\frac{4\sqrt{5}}{16} = \frac{\sqrt{5}}{16}.$$

4. $$\varepsilon\varphi\alpha\cdot\varepsilon\varphi 2\alpha\cdot\varepsilon\varphi 3\alpha\cdot\varepsilon\varphi 4\alpha = 1.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\varepsilon\varphi\alpha\cdot\varepsilon\varphi 2\alpha\cdot\varepsilon\varphi 3\alpha\cdot\varepsilon\varphi 4\alpha = \varepsilon\varphi 18^0\cdot\varepsilon\varphi 36^0\cdot\varepsilon\varphi 54^0\cdot\varepsilon\varphi 72^0 =$$

$$= \frac{\sqrt{25-10\sqrt{5}}}{5}\cdot\sqrt{5-2\sqrt{5}}\cdot\frac{1}{\sqrt{5-2\sqrt{5}}}\cdot\sqrt{5+2\sqrt{5}} =$$

$$= \frac{\sqrt{25-10\sqrt{5}}}{5\sqrt{5-2\sqrt{5}}}\cdot\sqrt{25-20} = \frac{\sqrt{25-10\sqrt{5}}\cdot\sqrt{5}}{5\sqrt{5-2\sqrt{5}}}\cdot\frac{\sqrt{25-10\sqrt{5}}}{\sqrt{25-10\sqrt{5}}} = 1.$$

49. Νὰ ἁπλοποιηθοῦν αἱ παραστάσεις :

1. $$E=3-4\sigma\upsilon\nu 2\alpha+\sigma\upsilon\nu 4\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\begin{aligned}E&=3-4\sigma\upsilon\nu 2\alpha+\sigma\upsilon\nu 4\alpha=3-4\sigma\upsilon\nu 2\alpha+\sigma\upsilon\nu 2(2\alpha)=\\&=3-4\sigma\upsilon\nu 2\alpha+2\sigma\upsilon\nu^2 2\alpha-1=2\sigma\upsilon\nu^2 2\alpha-4\sigma\upsilon\nu 2\alpha+2=\\&=2(\sigma\upsilon\nu^2 2\alpha-2\sigma\upsilon\nu 2\alpha+1)=2(\sigma\upsilon\nu 2\alpha-1)^2=2(1-2\eta\mu^2\alpha-1)^2=\\&=2(-2\eta\mu^2\alpha)^2=8\eta\mu^4\alpha.\end{aligned}$$

2. $$E_1=\frac{\eta\mu 4\alpha+\eta\mu 2\alpha}{1+\sigma\upsilon\nu 4\alpha+\sigma\upsilon\nu 2\alpha}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἄν $\alpha\neq(2k+1)\frac{\pi}{4}$ καὶ $\alpha+k_1\pi\pm\frac{\pi}{3}$, $(k,k_1)\in\mathbf{Z}$

$$E_1=\frac{\eta\mu 4\alpha+\eta\mu 2\alpha}{1+\sigma\upsilon\nu 4\alpha+\sigma\upsilon\nu 2\alpha}=\frac{2\eta\mu 2\alpha\sigma\upsilon\nu 2\alpha+\eta\mu 2\alpha}{1+2\sigma\upsilon\nu^2 2\alpha-1+\sigma\upsilon\nu 2\alpha}=\frac{\eta\mu 2\alpha(2\sigma\upsilon\nu 2\alpha+1)}{\sigma\upsilon\nu 2\alpha(2\sigma\upsilon\nu 2\alpha+1)}=$$

$$=\frac{\eta\mu 2\alpha}{\sigma\upsilon\nu 2\alpha}=\varepsilon\varphi 2\alpha.$$

3. $$E_2=4(\sigma\upsilon\nu^6\alpha+\eta\mu^6\alpha)-3(\sigma\upsilon\nu^4\alpha-\eta\mu^4\alpha)^2.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$E_3=4\,\sigma\upsilon\nu^6\alpha+\eta\mu^6\alpha)-3(\sigma\upsilon\nu^4\alpha-\eta\mu^4\alpha)^2\equiv$$

$$\begin{aligned}&\equiv 4(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)(\sigma\upsilon\nu^4\alpha+\eta\mu^4\alpha-\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha)-3(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)^2(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha)^2\equiv\\&\equiv 4[(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)^2-3\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha]-3(\sigma\upsilon\nu^2\alpha-\eta\mu^2\alpha)^2\equiv\\&\equiv 4(1-3\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha)-3(\sigma\upsilon\nu^4\alpha-2\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha+\eta\mu^4\alpha)\equiv\\&\equiv 4-12\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-3\sigma\upsilon\nu^4\alpha+6\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-3\eta\mu^4\alpha\equiv\\&\equiv 4-6\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-3\sigma\upsilon\nu^4\alpha-3\eta\mu^4\alpha\equiv 4-6\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-3(\sigma\upsilon\nu^4\alpha+\eta\mu^4\alpha)\equiv\\&\equiv 4-6\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-3[(\sigma\upsilon\nu^2\alpha+\eta\mu^2\alpha)^2-2\sigma\upsilon\nu^2\alpha\eta\mu^2\alpha]\equiv\\&\equiv 4-6\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-3+6\sigma\upsilon\nu^2\alpha\eta\mu^2\alpha\equiv 1.\end{aligned}$$

50. Νὰ ἀποδειχθοῦν αἱ ἀκόλουθοι ἰσότητες.

1. $$\frac{\sigma\varphi\frac{\theta}{2}+1}{\sigma\varphi\frac{\theta}{2}-1}=\frac{\sigma\upsilon\nu\theta}{1-\eta\mu\theta}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἄν $\theta\neq 2k\pi+\frac{\pi}{2}$, $k\in\mathbf{Z}$.

$$\frac{\sigma\varphi\frac{\theta}{2}+1}{\sigma\varphi\frac{\theta}{2}-1}=\frac{\sigma\upsilon\nu\frac{\theta}{2}+\eta\mu\frac{\theta}{2}}{\sigma\upsilon\nu\frac{\theta}{2}-\eta\mu\frac{\theta}{2}}=\frac{\sigma\upsilon\nu^2\frac{\theta}{2}-\eta\mu^2\frac{\theta}{2}}{\left(\sigma\upsilon\nu\frac{\theta}{2}-\eta\mu\frac{\theta}{2}\right)^2}=$$

$$=\frac{\sigma\upsilon\nu\theta}{1-2\sigma\upsilon\nu\frac{\theta}{2}\eta\mu\frac{\theta}{2}}=\frac{\sigma\upsilon\nu\theta}{1-\eta\mu\theta}.$$

2. $$\textbf{τεμα}-\textbf{εφα}=\textbf{εφ}\left(45^0-\frac{\alpha}{2}\right),$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq \pm \dfrac{\pi}{2}$ καὶ $\alpha \neq 2k_1\pi+\pi$.

$$τεμα-εφα=\frac{1}{συνα}-\frac{ημα}{συνα}=\frac{1-ημα}{συνα}=\frac{1-2ημ\frac{\alpha}{2}\,συν\frac{\alpha}{2}}{συν^2\frac{\alpha}{2}-ημ^2\frac{\alpha}{2}}=$$

$$=\frac{συν^2\frac{\theta}{2}+ημ^2\frac{\theta}{2}-2ημ\frac{\alpha}{2}\,συν\frac{\alpha}{2}}{συν^2\frac{\alpha}{2}-ημ^2\frac{\alpha}{2}}=\frac{\left(συν\frac{\alpha}{2}-ημ\frac{\alpha}{2}\right)^2}{συν^2\frac{\alpha}{2}-ημ^2\frac{\alpha}{2}}=$$

$$=\frac{συν\frac{\alpha}{2}-ημ\frac{\alpha}{2}}{συν\frac{\alpha}{2}+ημ\frac{\alpha}{2}}=\frac{1-εφ\frac{\alpha}{2}}{1+εφ\frac{\alpha}{2}}=\frac{εφ45^0-εφ\frac{\alpha}{2}}{1+εφ45^0\cdot εφ\frac{\alpha}{2}}=εφ\left(45^0-\frac{\alpha}{2}\right).$$

3. $$\textbf{εφα}+\textbf{τεμα}=\textbf{σφ}\left(45^0-\frac{\alpha}{2}\right).$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq k\pi+\dfrac{\pi}{2}$, $\alpha \neq 2k_1\pi$, $(k, k_1) \in \mathbf{Z}$.

$$εφα+τεμα=\frac{ημα}{συνα}+\frac{1}{συνα}=\frac{1+ημα}{συνα}=\frac{συν^2\frac{\alpha}{2}+ημ^2\frac{\alpha}{2}+2ημ\frac{\alpha}{2}\,συν\frac{\alpha}{2}}{συν^2\frac{\alpha}{2}-ημ^2\frac{\alpha}{2}}=$$

$$=\frac{\left(συν\frac{\alpha}{2}+ημ\frac{\alpha}{2}\right)^2}{συν^2\frac{\alpha}{2}-ημ^2\frac{\alpha}{2}}=\frac{συν\frac{\alpha}{2}+ημ\frac{\alpha}{2}}{συν\frac{\alpha}{2}-ημ\frac{\alpha}{2}}=\frac{σφ\frac{\alpha}{2}+1}{σφ\frac{\alpha}{2}-1}=$$

$$=\frac{σφ45^0\cdot σφ\frac{\alpha}{2}+1}{σφ\frac{\alpha}{2}-σφ45^0}=σφ\left(45^0-\frac{\alpha}{2}\right).$$

4. $$\frac{\textbf{1}+\textbf{συνα}+\textbf{συν}\frac{\alpha}{2}}{\textbf{ημα}+\textbf{ημα}\frac{\alpha}{2}}=\textbf{σφ}\frac{\alpha}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha \neq 2k\pi$, $\alpha \neq 4k_1\pi+\dfrac{4\pi}{3}$, $(k, k_1) \in \mathbf{Z}$.

$$\frac{1+συνα+συν\frac{\alpha}{2}}{ημα+ημ\frac{\alpha}{2}}=\frac{1+2συν^2\frac{\alpha}{2}-1+συν\frac{\alpha}{2}}{2ημ\frac{\alpha}{2}\,συν\frac{\alpha}{2}+ημ\frac{\alpha}{2}}=$$

$$=\frac{\sigma\upsilon\nu\frac{\alpha}{2}\left(2\sigma\upsilon\nu\frac{\alpha}{2}+1\right)}{\eta\mu\frac{\alpha}{2}\left(2\sigma\upsilon\nu\frac{\alpha}{2}+1\right)}=\frac{\sigma\upsilon\nu\frac{\alpha}{2}}{\eta\mu\frac{\alpha}{2}}=\sigma\varphi\frac{\alpha}{2}.$$

5. $$\frac{\eta\mu 2\alpha}{1-\sigma\upsilon\nu 2\alpha}\cdot\frac{1-\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\alpha}=\varepsilon\varphi\frac{\alpha}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\pi$, $k\in\mathbf{Z}$ καὶ $\alpha\neq k_1\pi+\frac{\pi}{2}$, $k_1\in\mathbf{Z}$

$$\frac{\eta\mu 2\alpha}{1-\sigma\upsilon\nu 2\alpha}\cdot\frac{1-\sigma\upsilon\nu\alpha}{\sigma\upsilon\nu\alpha}=\frac{2\eta\mu\alpha\sigma\upsilon\nu\alpha}{1-1+2\eta\mu^2\alpha}\cdot\frac{1-1+2\eta\mu^2\frac{\alpha}{2}}{\sigma\upsilon\nu\alpha}=$$

$$=\frac{1}{\eta\mu\alpha}\cdot 2\eta\mu^2\frac{\alpha}{2}=\frac{1}{2\eta\mu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}\cdot 2\eta\mu^2\frac{\alpha}{2}=\frac{\eta\mu\frac{\alpha}{2}}{\sigma\upsilon\nu\frac{\alpha}{2}}=\varepsilon\varphi\frac{\alpha}{2}.$$

6. $$\frac{\eta\mu 2\alpha}{1+\sigma\upsilon\nu 2\alpha}\cdot\frac{\sigma\upsilon\nu\alpha}{1+\sigma\upsilon\nu\alpha}=\varepsilon\varphi\frac{\alpha}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\pi\pm\frac{\pi}{2}$, $\alpha\neq 2k_1\pi\pm\pi$, $(k,k_1)\in\mathbf{Z}$.

$$\frac{\eta\mu 2\alpha}{1+\sigma\upsilon\nu 2\alpha}\cdot\frac{\sigma\upsilon\nu\alpha}{1+\sigma\upsilon\nu\alpha}=\frac{2\eta\mu\alpha\cdot\sigma\upsilon\nu\alpha}{1+2\sigma\upsilon\nu^2\alpha-1}\cdot\frac{\sigma\upsilon\nu\alpha}{1+2\sigma\upsilon\nu^2\frac{\alpha}{2}-1}=\frac{\eta\mu\alpha}{\sigma\upsilon\nu\alpha}\cdot\frac{\sigma\upsilon\nu\alpha}{2\sigma\upsilon\nu\frac{\alpha}{2}}=$$

$$=\frac{2\eta\mu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}{2\sigma\upsilon\nu^2\frac{\alpha}{2}}=\frac{\eta\mu\frac{\alpha}{2}}{\sigma\upsilon\nu\frac{\alpha}{2}}=\varepsilon\varphi\frac{\alpha}{2}.$$

7. $$\sigma\varphi\frac{\alpha}{2}-\varepsilon\varphi\frac{\alpha}{2}=2\sigma\varphi\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq 2k\pi$, $\alpha\neq k_1\pi+\frac{\pi}{2}$, $(k,k_1)\in\mathbf{Z}$.

$$\sigma\varphi\frac{\alpha}{2}-\varepsilon\varphi\frac{\alpha}{2}=\frac{\sigma\upsilon\nu\frac{\alpha}{2}}{\eta\mu\frac{\alpha}{2}}-\frac{\eta\mu\frac{\alpha}{2}}{\sigma\upsilon\nu\frac{\alpha}{2}}=\frac{\sigma\upsilon\nu^2\frac{\alpha}{2}-\eta\mu^2\frac{\alpha}{2}}{\eta\mu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}=\frac{\sigma\upsilon\nu\alpha}{\eta\mu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}=$$

$$=\frac{2\sigma\upsilon\nu\alpha}{2\eta\mu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}=\frac{2\sigma\upsilon\nu\alpha}{\eta\mu\alpha}=2\sigma\varphi\alpha.$$

8. $$\varepsilon\varphi\left(45^\circ+\frac{\alpha}{2}\right)=\sqrt{\frac{1+\eta\mu\alpha}{1-\eta\mu\alpha}}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq\frac{\pi}{2}+k\pi$ καὶ $\alpha\neq 2k_1\pi+\frac{\pi}{2}$, $(k,k_1)\in\mathbf{Z}$

$$\varepsilon\varphi\left(45^0+\frac{\alpha}{2}\right)=\frac{\varepsilon\varphi 45^0+\varepsilon\varphi\frac{\alpha}{2}}{1-\varepsilon\varphi 45^0\cdot\varepsilon\varphi\frac{\alpha}{2}}=\frac{1+\varepsilon\varphi\frac{\alpha}{2}}{1-\varepsilon\varphi\frac{\alpha}{2}}=\frac{\sigma\upsilon\nu\frac{\alpha}{2}+\eta\mu\frac{\alpha}{2}}{\sigma\upsilon\nu\frac{\alpha}{2}-\eta\mu\frac{\alpha}{2}}=$$

$$=\frac{\left(\sigma\upsilon\nu\frac{\alpha}{2}+\eta\mu\frac{\alpha}{2}\right)^2}{\sigma\upsilon\nu^2\frac{\alpha}{2}-\eta\mu^2\frac{\alpha}{2}}=\frac{\sigma\upsilon\nu^2\frac{\alpha}{2}+\eta\mu^2\frac{\alpha}{2}+2\eta\mu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}{\sigma\upsilon\nu\alpha}=$$

$$=\frac{1+\eta\mu\alpha}{\sigma\upsilon\nu\alpha}=\sqrt{\frac{(1+\eta\mu\alpha)^2}{\sigma\upsilon\nu^2\alpha}}=\sqrt{\frac{(1+\eta\mu\alpha)^2}{1-\eta\mu^2\alpha}}=\sqrt{\frac{1+\eta\mu\alpha}{1-\eta\mu\alpha}}.$$

51. Νὰ ἀποδειχθῇ ὅτι :

1. $$(\sigma\upsilon\nu\alpha+\sigma\upsilon\nu\beta)^2+(\eta\mu\alpha-\eta\mu\beta)^2=4\sigma\upsilon\nu^2\frac{\alpha+\beta}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\sigma\upsilon\nu\alpha+\sigma\upsilon\nu\beta)^2+(\eta\mu\alpha-\eta\mu\beta)^2=\sigma\upsilon\nu^2\alpha+\sigma\upsilon\nu^2\beta+2\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu^2\alpha+$$
$$+\eta\mu^2\beta-2\eta\mu\alpha\eta\mu\beta=2+2(\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta-\eta\mu\alpha\eta\mu\beta)=2+2\sigma\upsilon\nu(\alpha+\beta)=$$
$$=2[1+\sigma\upsilon\nu(\alpha+\beta)]=2\left[1+2\sigma\upsilon\nu^2\frac{\alpha+\beta}{2}-1\right]=4\sigma\upsilon\nu^2\frac{\alpha+\beta}{2}.$$

2. $$(\sigma\upsilon\nu\alpha+\sigma\upsilon\nu\beta)^2+(\eta\mu\alpha+\eta\mu\beta)^2=4\sigma\upsilon\nu^2\frac{\alpha-\beta}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\sigma\upsilon\nu\alpha+\sigma\upsilon\nu\beta)^2+(\eta\mu\alpha+\eta\mu\beta)^2=\sigma\upsilon\nu^2\alpha+\sigma\upsilon\nu^2\beta+2\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu^2\alpha+$$
$$+\eta\mu^2\beta+2\eta\mu\alpha\eta\mu\beta=2+2(\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu\alpha\eta\mu\beta)=2+2\sigma\upsilon\nu(\alpha-\beta)=$$
$$=2[1+\sigma\upsilon\nu(\alpha-\beta)]=2\left[1+2\sigma\upsilon\nu^2\frac{\alpha-\beta}{2}-1\right]=4\sigma\upsilon\nu^2\frac{\alpha-\beta}{2}.$$

3. $$(\sigma\upsilon\nu\alpha-\sigma\upsilon\nu\beta)^2+(\eta\mu\alpha-\eta\mu\beta)^2=4\eta\mu^2\frac{\alpha-\beta}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\sigma\upsilon\nu\alpha-\sigma\upsilon\nu\beta)^2+(\eta\mu\alpha-\eta\mu\beta)^2=\sigma\upsilon\nu^2\alpha+\sigma\upsilon\nu^2\beta-2\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu^2\alpha+$$
$$+\eta\mu^2\beta-2\eta\mu\alpha\eta\mu\beta=2-2(\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta+\eta\mu\alpha\eta\mu\beta)=2-2\sigma\upsilon\nu(\alpha-\beta)=$$
$$=[1-\sigma\upsilon\nu(\alpha-\beta)]=2\left[1-1+2\eta\mu^2\frac{\alpha-\beta}{2}\right]=4\eta\mu^2\frac{\alpha-\beta}{2}.$$

5. $$\eta\mu^2\left(\frac{\pi}{8}+\frac{\alpha}{2}\right)-\eta\mu^2\left(\frac{\pi}{8}-\frac{\alpha}{2}\right)=\frac{\sqrt{2}}{2}\eta\mu\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς, βάσει τῆς ταυτότητος :

$$\eta\mu^2\alpha-\eta\mu^2\beta=\eta\mu(\alpha+\beta)\eta\mu(\alpha-\beta)$$

ὅτι : $$\eta\mu^2\left(\frac{\pi}{8}+\frac{\alpha}{2}\right)-\eta\mu^2\left(\frac{\pi}{8}-\frac{\alpha}{2}\right)=$$

$$=\eta\mu\left[\frac{\pi}{8}+\frac{\alpha}{2}+\frac{\pi}{8}-\frac{\alpha}{2}\right]\eta\mu\left[\frac{\pi}{8}+\frac{\alpha}{2}-\frac{\pi}{8}+\frac{\alpha}{2}\right]=$$

$$=\eta\mu\,\frac{\pi}{4}\,\eta\mu\alpha=\frac{\sqrt{2}}{2}\,\eta\mu\alpha.$$

52. Γνωστοῦ ὄντος ὅτι $\sigma\upsilon\nu 315^{0}=\frac{\sqrt{2}}{2}$, νὰ ὑπολογισθοῦν τὸ ημ (157° 30′) καὶ τὸ συν(157° 30′).

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu(157^{0}30')=+\sqrt{\frac{1-\sigma\upsilon\nu 315^{0}}{2}}=\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}}=\frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\sigma\upsilon\nu(157^{0}30')=-\sqrt{\frac{1+\sigma\upsilon\nu 315^{0}}{2}}=-\sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}=-\frac{\sqrt{2+\sqrt{2}}}{2}.$$

53. Νὰ ἀποδειχθῇ ὅτι:

1. $$\eta\mu\frac{\pi}{16}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\,\frac{\pi}{16}=\sqrt{\frac{1-\sigma\upsilon\nu\,\frac{\pi}{8}}{2}}=\sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$$

2. $$\sigma\upsilon\nu\,\frac{\pi}{16}=\sqrt{\frac{1+\sigma\upsilon\nu\frac{\pi}{8}}{2}}=\sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$$

3. $$\eta\mu\frac{\pi}{32}=\sqrt{\frac{1-\sigma\upsilon\nu\frac{\pi}{16}}{2}}=\sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}=$$

$$=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

4. $$\sigma\upsilon\nu\,\frac{\pi}{32}=\sqrt{\frac{1+\sigma\upsilon\nu\,\frac{\pi}{16}}{2}}=\sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}}=$$

$$=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

. . . .

54. Νὰ ἀποδειχθῇ ὅτι :

1. $$\eta\mu\frac{\pi}{24}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}.$$

Λύσις Ἔχομεν διαδοχικῶς:

$$\eta\mu\frac{\pi}{24}=\sqrt{\frac{1-\sigma\upsilon\nu\frac{\pi}{12}}{2}}=\sqrt{\frac{1-\frac{1}{2}\sqrt{2+\sqrt{3}}}{2}}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}\,.$$

2. $$\sigma\upsilon\nu\frac{\pi}{24}=\sqrt{\frac{1+\sigma\upsilon\nu\frac{\pi}{12}}{2}}=\sqrt{\frac{1+\frac{1}{2}\sqrt{2+\sqrt{3}}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}\,.$$

3. $$\eta\mu\frac{\pi}{48}=\frac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}\,.$$

4, $$\sigma\upsilon\nu\frac{\pi}{48}=\frac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}\,.$$

. . . .

55. Νὰ ἀποδειχθῇ ὅτι :

1. $$\eta\mu 9^0=\sigma\upsilon\nu 81^0=\frac{1}{4}\left(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right).$$

Λύσις. Ἐπειδὴ $81^0+9^0=90^0$, ἕπεται ὅτι:

$$\sigma\upsilon\nu 81^0=\eta\mu 9^0=\sqrt{\frac{1-\sigma\upsilon\nu 18^0}{2}}=\sqrt{\frac{1-\frac{1}{4}\sqrt{10+2\sqrt{5}}}{2}}=$$

$$=\sqrt{\frac{4-\sqrt{10+2\sqrt{5}}}{8}}=\frac{1}{4}\sqrt{8-2\sqrt{10+2\sqrt{5}}}=\frac{1}{4}\left(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}\right)\cdot$$

2. $$\sigma\upsilon\nu 9^0=\eta\mu 81^0=\frac{1}{4}\left(\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}\right).$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu 81^0=\sigma\upsilon\nu 9^0=\sqrt{\frac{1+\sigma\upsilon\nu 18^0}{2}}=\sqrt{\frac{1+\frac{1}{4}\sqrt{10+2\sqrt{5}}}{2}}=$$

$$=\sqrt{\frac{4+\sqrt{10+2\sqrt{5}}}{8}}=\frac{1}{4}\sqrt{8+2\sqrt{10+2\sqrt{5}}}=\frac{1}{4}\left(\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}\right).$$

56. Νὰ ἀποδειχθῇ ὅτι :

1. $\eta\mu 27^0=\sigma\upsilon\nu 63^0=\frac{1}{4}\left(\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}\right).$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu 63^0=\eta\mu 27^0=\sqrt{\frac{1-\sigma\upsilon\nu 54^0}{2}}=\sqrt{\frac{1-\frac{1}{4}\sqrt{10-2\sqrt{5}}}{2}}=\sqrt{\frac{4-\sqrt{10-2\sqrt{5}}}{8}}=$$

$$=\frac{1}{4}\sqrt{8-2\sqrt{10-2\sqrt{5}}}=\frac{1}{4}\left(\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}\right).$$

2. $\sigma\upsilon\nu 27^0=\eta\mu 63^0=\frac{1}{4}\left(\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}}\right)$

Ἔχομεν διαδοχικῶς :

$$\eta\mu 63^0=\sigma\upsilon\nu 27^0=\sqrt{\frac{1+\sigma\upsilon\nu 54^0}{2}}=\sqrt{\frac{1+\frac{1}{4}\sqrt{10-2\sqrt{5}}}{2}}=$$

$$=\sqrt{\frac{4+\sqrt{10-2\sqrt{5}}}{8}}=\frac{1}{4}\sqrt{8+2\sqrt{10-2\sqrt{5}}}=\frac{1}{4}\left(\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}}\right).$$

57. Γνωστοῦ ὄντος ὅτι: $48^0=18^0+30^0$ καὶ $3^0=48^0-45^0$, νὰ ἀποδειχθῇ ὅτι :

1. $\eta\mu 48^0=\sigma\upsilon\nu 42^0=\frac{1}{8}\sqrt{10+2\sqrt{2}}+\frac{\sqrt{3}}{8}(-1+\sqrt{5}).$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu 48^0=\eta\mu(18^0+30^0)=\eta\mu 18^0\sigma\upsilon\nu 30^0+\eta\mu 30^0\sigma\upsilon\nu 18^0=$$

$$=\frac{\sqrt{5}-1}{4}\cdot\frac{\sqrt{3}}{2}+\frac{1}{2}\cdot\frac{1}{4}\sqrt{10+2\sqrt{5}}=\frac{1}{8}\sqrt{10+2\sqrt{5}}+\frac{\sqrt{3}}{8}(-1+\sqrt{5}).$$

2. $\eta\mu 24^0=\sigma\upsilon\nu 66^0=\frac{\sqrt{3}}{8}(1+\sqrt{5})-\frac{1}{8}\sqrt{10+2\sqrt{5}}.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu 66^0=\eta\mu 24^0=\sqrt{\frac{1-\sigma\upsilon\nu 48^0}{2}}=\frac{\sqrt{3}}{8}(1+\sqrt{5})-\frac{1}{8}\sqrt{10+2\sqrt{5}}.$$

3. $\eta\mu 12^0=\sigma\upsilon\nu 78^0=\frac{1}{8}\sqrt{10+2\sqrt{5}}-\frac{\sqrt{3}}{8}(-1+\sqrt{5}).$

4. $\eta\mu 6^0=\sigma\upsilon\nu 84^0=\frac{\sqrt{3}}{8}(10-2\sqrt{5})-\frac{1}{8}(1+\sqrt{5}).$

58. Νὰ ἀποδειχθῇ ὅτι :

1. $$\text{συν}^4 \frac{\pi}{8} + \text{συν}^4 \frac{3\pi}{8} = \frac{3}{4}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{συν}^4 \frac{\pi}{8} + \text{συν}^4 \frac{3\pi}{8} = \left(\frac{1+\text{συν}\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1+\text{συν}\frac{3\pi}{4}}{2}\right)^2 =$$

$$= \left(\frac{1+\frac{\sqrt{2}}{2}}{2}\right)^2 + \left(\frac{1-\frac{\sqrt{2}}{2}}{2}\right)^2 = \frac{(2+\sqrt{2})^2}{16} + \frac{(2-\sqrt{2})^2}{16} =$$

$$= \frac{4+2+4\sqrt{2}+4+2-4\sqrt{2}}{16} = \frac{12}{16} = \frac{3}{4}.$$

2. $$\text{ημ}^4 \frac{\pi}{8} + \text{ημ}^4 \frac{3\pi}{8} = \frac{3}{4}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{ημ}^4 \frac{\pi}{8} + \text{ημ}^4 \frac{3\pi}{8} = \left(\frac{1-\text{συν}\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1-\text{συν}\frac{3\pi}{4}}{2}\right)^2 =$$

$$= \left(\frac{1-\frac{\sqrt{2}}{2}}{2}\right)^2 + \left(\frac{1+\frac{\sqrt{2}}{2}}{2}\right)^2 = \frac{(2-\sqrt{2})^2}{16} + \frac{(2+\sqrt{2})^2}{16} = \frac{3}{4}.$$

3. $$\text{συν}^4 \frac{\pi}{8} + \text{συν}^4 \frac{3\pi}{8} + \text{συν}^4 \frac{5\pi}{8} + \text{συν}^4 \frac{7\pi}{8} = \frac{3}{2}.$$

Λύσις. Ἔχομεν: $\text{συν} \frac{\pi}{8} = -\text{συν} \frac{7\pi}{8}$, καθόσον $\frac{\pi}{8} + \frac{7\pi}{8} = \pi$

καὶ $\text{συν} \frac{3\pi}{8} = -\text{συν} \frac{5\pi}{8}$, καθόσον $\frac{3\pi}{8} + \frac{5\pi}{8} = \pi$.

$$\text{συν}^4 \frac{\pi}{8} + \text{συν}^4 \frac{3\pi}{8} + \text{συν}^4 \frac{5\pi}{8} + \text{συν}^4 \frac{7\pi}{2} = 2\text{συν}^4 \frac{\pi}{8} + 2\text{συν}^4 \frac{3\pi}{8} =$$

$$= 2\left[\left(\frac{1+\text{συν}\frac{\pi}{4}}{2}\right)^2 + \left(\frac{1-\text{συν}\frac{3\pi}{4}}{2}\right)^2\right] = 2\left[\left(\frac{1+\frac{\sqrt{2}}{2}}{2}\right)^2 + \left(\frac{1-\frac{\sqrt{2}}{2}}{2}\right)^2\right] =$$

$$= 2\left[\frac{(2+\sqrt{2})^2}{16} + \frac{(2-\sqrt{2})^2}{16}\right] = \frac{4+2+4\sqrt{2}+4+2-4\sqrt{2}}{8} = \frac{12}{8} = \frac{3}{2}.$$

5. $$\text{συν}^4\theta + \text{συν}^4\left(\frac{\pi}{4}+\theta\right) + \text{συν}^4\left(\frac{\pi}{2}+\theta\right) + \text{συν}^4\left(\frac{3\pi}{4}+\theta\right) = \frac{3}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν^4θ+συν^4\left(\frac{π}{4}+θ\right)+συν^4\left(\frac{π}{2}+θ\right)+συν^4\left(\frac{3π}{4}+θ\right)=$$

$$=\left(\frac{1+συν2θ}{2}\right)^2+\left[\frac{1+συν\left(\frac{π}{2}+2θ\right)}{2}\right]^2+\left[\frac{1+συν(π+2θ)}{2}\right]^2+$$

$$=\left[\frac{1+συν\left(\frac{3π}{2}+2θ\right)}{2}\right]^2=\left(\frac{1+συν2θ}{2}\right)^2+\left(\frac{1-ημ2θ}{2}\right)^2+$$

$$+\left(\frac{1-συν2θ}{2}\right)^2+\left(\frac{1+ημ2θ}{2}\right)^2=$$

$$=\frac{(1+συν2θ)^2+(1-ημ2θ)^2+(1-συν2θ)^2+(1+ημ2θ)^2}{4}=$$

$$=\frac{1+συν^22θ+2συν2θ+1+ημ^22θ-2ημ2θ+1+συν^22θ-2συν2θ+1+ημ^22θ+2ημ2θ}{4}=$$

$$=\frac{4+1+1}{4}=\frac{6}{4}=\frac{3}{2}.$$

5. $\left(\mathbf{1+συν\frac{π}{8}}\right)\left(\mathbf{1+συν\frac{3π}{8}}\right)\left(\mathbf{1+συν\frac{5π}{8}}\right)\left(\mathbf{1+συν\frac{7π}{8}}\right)=\mathbf{\frac{1}{8}}.$

Λύσις. Ἐπειδὴ $\frac{7π}{8}+\frac{π}{8}=π \Longrightarrow συν\frac{7π}{8}=-συν\frac{π}{8}$

καὶ $\frac{5π}{8}+\frac{3π}{8}=π \Longrightarrow συν\frac{5π}{8}=-συν\frac{3π}{8}$. Ἄρα:

$$\left(1+συν\frac{π}{8}\right)\left(1+συν\frac{3π}{8}\right)\left(1+συν\frac{5π}{8}\right)\left(1+συν\frac{7π}{8}\right)=$$

$$=\left(1+συν\frac{π}{8}\right)\left(1+συν\frac{3π}{8}\right)\left(1-συν\frac{3π}{8}\right)\left(1-συν\frac{π}{8}\right)=$$

$$=\left(1-συν^2\frac{π}{8}\right)\left(1-συν^2\frac{3π}{8}\right)=ημ^2\frac{π}{8}\,ημ^2\frac{3π}{8}=$$

$$=\frac{1-συν\frac{π}{4}}{2}\cdot\frac{1-συν\frac{3π}{4}}{2}=\frac{1-\frac{\sqrt{2}}{2}}{2}\cdot\frac{1+\frac{\sqrt{2}}{2}}{2}=$$

$$=\frac{2-\sqrt{2}}{4}\cdot\frac{2+\sqrt{2}}{4}=\frac{4-2}{16}=\frac{2}{16}=\frac{1}{8}.$$

59. Ἐὰν $\mathbf{συνx=\frac{α}{β+γ}}$, $\mathbf{συνy=\frac{β}{γ+α}}$, $\mathbf{συνω=\frac{γ}{α+β}}$, **νὰ ἀποδειχθῇ ὅτι:**

$$\mathbf{εφ^2\frac{x}{2}+εφ^2\frac{y}{2}+εφ^2\frac{ω}{2}=1.}$$

Λύσις. Ἔχομεν: $1+\text{συν}x=1+\dfrac{\alpha}{\beta+\gamma}=\dfrac{\alpha+\beta+\gamma}{\beta+\gamma}$

καὶ ὁμοίως: $1+\text{συν}y=\dfrac{\alpha+\beta+\gamma}{\gamma+\alpha}$ καὶ $1+\text{συν}\omega=\dfrac{\alpha+\beta+\gamma}{\alpha+\beta}$.

καὶ $1-\text{συν}x=1-\dfrac{\alpha}{\beta+\gamma}=\dfrac{\beta+\gamma-\alpha}{\beta+\gamma}$, $1-\text{συν}y=\dfrac{\gamma+\alpha-\beta}{\gamma+\alpha}$, $1-\text{συν}\omega=\dfrac{\alpha+\beta-\gamma}{\alpha+\beta}$

$$\text{εφ}^2\frac{x}{2}=\frac{1-\text{συν}x}{1+\text{συν}x}=\frac{\dfrac{\beta+\gamma-\alpha}{\beta+\gamma}}{\dfrac{\alpha+\beta+\gamma}{\beta+\gamma}}=\frac{\beta+\gamma-\alpha}{\alpha+\beta+\gamma},$$

καὶ ὁμοίως: $\text{εφ}^2\dfrac{y}{2}=\dfrac{\gamma+\alpha-\beta}{\alpha+\beta+\gamma}$, $\text{εφ}^2\dfrac{\omega}{2}=\dfrac{\alpha+\beta-\gamma}{\alpha+\beta+\gamma}$. Ἄρα:

$$\text{εφ}^2\frac{x}{2}+\text{εφ}^2\frac{y}{2}+\text{εφ}^2\frac{\omega}{2}=\frac{\beta+\gamma-\alpha}{\alpha+\beta+\gamma}+\frac{\gamma+\alpha-\beta}{\alpha+\beta+\gamma}+\frac{\alpha+\beta-\gamma}{\alpha+\beta-\gamma}=\frac{\alpha+\beta+\gamma}{\alpha+\beta+\gamma}=1.$$

60. Ἐὰν $\text{συν}\alpha+\text{συν}\beta+\text{συν}\gamma=0$, νὰ ὑπολογισθῇ ἡ τιμὴ τῆς παραστάσεως:

$$K=\frac{\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}{\text{συν}3\alpha+\text{συν}3\beta+\text{συν}3\gamma}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$K=\frac{\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}{\text{συν}3\alpha+\text{συν}3\beta+\text{συν}3\gamma}=\frac{\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}{4\text{συν}^3\alpha-3\text{συν}\alpha+4\text{συν}^3\beta-3\text{συν}\beta+4\text{συν}^3\gamma-3\text{συν}\gamma}$$

$$=\frac{\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}{4(\text{συν}^3\alpha+\text{συν}^3\beta+\text{συν}^3\gamma)-3(\text{συν}\alpha+\text{συν}\beta+\text{συν}\gamma)}=\frac{\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}{4(\text{συν}^3\alpha+\text{συν}^3\beta+\text{συν}^3\gamma)}$$

Ἐπειδὴ δὲ $\text{συν}^3\alpha+\text{συν}^3\beta+\text{συν}^3\gamma=3\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma$ (λόγῳ τῆς ὑποθέσεως καὶ τῆς γνωστῆς ταυτότητος τῆς Ἀλγέβρας), ἕπεται ὅτι:

$$K=\frac{\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}{4\cdot 3\text{συν}\alpha\,\text{συν}\beta\,\text{συν}\gamma}=\frac{1}{12}.$$

61. Ἐὰν $\text{ημ}x+\text{ημ}y+\text{ημ}\omega=0$, νὰ ὑπολογισθῇ ἡ τιμὴ τῆς παραστάσεως:

$$\Lambda=\frac{\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega}{\text{ημ}3x+\text{ημ}3y+\text{ημ}3\omega}.$$

Λύσις. Ἐπειδὴ $\text{ημ}x+\text{ημ}y+\text{ημ}\omega=0$, ἕπεται ὅτι:

$$\text{ημ}^3x+\text{ημ}^3y+\text{ημ}^3\omega=3\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega.$$

Θὰ ἔχωμεν διαδοχικῶς:

$$\Lambda\equiv\frac{\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega}{\text{ημ}3x+\text{ημ}3y+\text{ημ}3\omega}=\frac{\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega}{3\text{ημ}x-4\text{ημ}^3x+3\text{ημ}y-4\text{ημ}^3y+3\text{ημ}\omega-4\text{ημ}^3\omega}=$$

$$=\frac{\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega}{3(\text{ημ}x+\text{ημ}y+\text{ημ}\omega)-4(\text{ημ}^3x+\text{ημ}^3y+\text{ημ}^3\omega)}=$$

$$=\frac{\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega}{-4\cdot 3\text{ημ}x\,\text{ημ}y\,\text{ημ}\omega}=-\frac{1}{12}.$$

62. Νὰ ἀποδειχθῇ ὅτι :

$$\varepsilon\varphi\left(\alpha-\beta+\frac{\pi}{3}\right)+\varepsilon\varphi\left(\beta-\gamma+\frac{\pi}{3}\right)+\varepsilon\varphi\left(\gamma-\alpha+\frac{\pi}{3}\right)=$$

$$=\varepsilon\varphi\left(\alpha-\beta+\frac{\pi}{3}\right)\varepsilon\varphi\left(\beta-\gamma+\frac{\pi}{3}\right)\varepsilon\varphi\left(\gamma-\alpha+\frac{\pi}{3}\right).$$

Ἀπόδειξις. Θέτομεν :

$$\left.\begin{aligned}\alpha-\beta+\frac{\pi}{3}&=x\\ \beta-\gamma+\frac{\pi}{3}&=y\\ \gamma-\alpha+\frac{\pi}{3}&=\omega\end{aligned}\right\}\Rightarrow x+y+\omega=\pi \Rightarrow x+y=\pi-\omega \quad \text{ἢ} \quad \varepsilon\varphi(x+y)=-\varepsilon\varphi\omega$$

ἢ $$\frac{\varepsilon\varphi x+\varepsilon\varphi y}{1-\varepsilon\varphi x\varepsilon\varphi y}=-\varepsilon\varphi\omega \quad \text{ἢ} \quad \varepsilon\varphi x+\varepsilon\varphi y+\varepsilon\varphi\omega=\varepsilon\varphi x\varepsilon\varphi y\varepsilon\varphi\omega$$ ἢ

$$\varepsilon\varphi\left(\alpha-\beta+\frac{\pi}{3}\right)+\varepsilon\varphi\left(\beta-\gamma+\frac{\pi}{3}\right)+\varepsilon\varphi\left(\gamma-\alpha+\frac{\pi}{3}\right)=$$

$$=\varepsilon\varphi\left(\alpha-\beta+\frac{\pi}{3}\right)\varepsilon\varphi\left(\beta-\gamma+\frac{\pi}{3}\right)\varepsilon\varphi\left(\gamma-\alpha+\frac{\pi}{3}\right).$$

63. Ἐὰν $\sigma\upsilon\nu(\alpha-\beta)\eta\mu(\gamma-\delta)=\sigma\upsilon\nu(\alpha+\beta)\eta\mu(\gamma+\delta)$, **τότε :**

$$\sigma\varphi\delta=\sigma\varphi\alpha\sigma\varphi\beta\sigma\varphi\gamma.$$

***Λύσις*:** Ἡ δοθεῖσα σχέσις γράφεται :

$$\frac{\sigma\upsilon\nu(\alpha-\beta)}{\sigma\upsilon\nu(\alpha+\beta)}=\frac{\eta\mu(\gamma+\delta)}{\eta\mu(\gamma-\delta)} \quad \text{ἢ} \quad \frac{\sigma\upsilon\nu(\alpha-\beta)+\sigma\upsilon\nu(\alpha+\beta)}{\sigma\upsilon\nu(\alpha-\beta)-\sigma\upsilon\nu(\alpha+\beta)}=$$

$$=\frac{\eta\mu(\gamma+\delta)+\eta\mu(\gamma-\delta)}{\eta\mu(\gamma+\delta)-\eta\mu(\gamma-\delta)} \quad \text{ἢ} \quad \frac{2\sigma\upsilon\nu\alpha\sigma\upsilon\nu\beta}{2\eta\mu\alpha\eta\mu\beta}=\frac{2\eta\mu\gamma\sigma\upsilon\nu\delta}{2\eta\mu\delta\sigma\upsilon\nu\gamma}$$

ἢ $$\sigma\varphi\alpha\sigma\varphi\beta=\sigma\varphi\delta\cdot\varepsilon\varphi\gamma=\sigma\varphi\delta\cdot\frac{1}{\sigma\varphi\gamma}$$

ἐξ οὗ : $$\sigma\varphi\delta=\sigma\varphi\alpha\sigma\varphi\beta\sigma\varphi\gamma.$$

64. Ἐὰν $\alpha\eta\mu\omega\eta\mu\varphi\pm\beta\sigma\upsilon\nu\omega\sigma\upsilon\nu\varphi=0$, **νὰ δειχθῇ ὅτι ἡ παράστασις :**

$$K=\frac{1}{\alpha\eta\mu^2\omega+\beta\sigma\upsilon\nu^2\omega}+\frac{1}{\alpha\eta\mu^2\varphi+\beta\sigma\upsilon\nu^2\varphi}$$

εἶναι ἀνεξάρτητος τῶν ω καὶ φ, ἂν $\alpha\beta\neq0$ **καὶ** $\alpha\neq\beta$.

Ἀπόδειξις. Ἐὰν $\sigma\upsilon\nu\omega=0$, τότε $\eta\mu\omega=\pm1$, ὁπότε $\eta\mu\varphi=0$, ἐξ οὗ $\sigma\upsilon\nu\varphi=\pm1$, καὶ ἡ παράστασις γίνεται :

$$K=\frac{1}{\alpha}+\frac{1}{\beta}.$$

Ἔχομεν διαδοχικῶς :

$$Ε-\left(\frac{1}{α}+\frac{1}{β}\right)=\frac{1}{αημ^2ω+βσυν^2ω}-\frac{1}{α}+\frac{1}{αημ^2φ+βσυν^2φ}-\frac{1}{β}=$$

$$=\frac{(α-β)συν^2ω}{α(αημ^2ω+βσυν^2ω)}-\frac{(α-β)ημ^2φ}{β(αημ^2φ+βσυν^2φ)}=$$

$$=(α-β)\left[\frac{συν^2ω}{α(αημ^2ω+βσυν^2ω)}-\frac{ημ^2φ}{β(αημ^2φ+βσυν^2φ)}\right]=$$

$$=\frac{α-β}{αβ}\cdot\frac{β(αημ^2φ+βσυν^2φ)συν^2ω-α(αημ^2ω+βσυν^2ω)ημ^2φ}{(αημ^2ω+βσυν^2ω)(αημ^2φ+βσυν^2φ)}=$$

$$=\frac{α-β}{αβ}\cdot\frac{β^2συν^2φσυν^2ω-α^2ημ^2φημ^2ω}{(αημ^2ω+βσυν^2ω)(αημ^2φ+βσυν^2φ)}=\frac{α-β}{αβ}\cdot\frac{0}{»}=0,$$

Ἄρα:

$$Ε=\frac{1}{α}+\frac{1}{β}.$$

Ἡ παράστασις μένει πάλιν ἀμετάβλητος, ἂν $α=β\neq 0$.

65. Ἐὰν $0<(α,β,γ)<\frac{π}{2}$, νὰ ἀποδειχθῇ ὅτι :

$$\mathbf{ημ(α+β+γ)<ημα+ημβ+ημγ.}$$

Ἀπόδειξις. Ἐπειδή :

$$\left.\begin{array}{l}0<α<\frac{π}{2}\\0<β<\frac{π}{2}\\0<γ<\frac{π}{2}\end{array}\right\}\Longrightarrow\left.\begin{array}{l}συνα<1\\συνβ<1\\συνγ<1\end{array}\right\}\text{ καὶ }\left.\begin{array}{l}ημα>0\\ημβ>0\\ημγ>0\end{array}\right\}\Longrightarrow\left.\begin{array}{l}συνασυνβ<1\\συνβσυνγ<1\\συνγσυνα<1\end{array}\right\}\Longrightarrow\begin{array}{l}ημγσυνασυνβ<ημγ\\ημασυνβσυνγ<ημα\\ημβσυνγσυνα<ημβ\end{array}$$

Ἄρα : $ημασυνβσυνγ+ημβσυνγσυνα+ημγσυνασυνβ<ημα+ημβ+ημγ.$ (1)

Ἀλλὰ $ημαημβημγ>0$ ἢ $-ημαημβημγ<0$ (2)

Ἄρα: $ημασυνβσυνγ+ημβσυνγσυνα+ημγσυνασυνβ-ημαημβημγ<ημα+ημβ+ημγ$

ἢ $ημ(α+β+γ)<ημα+ημβ+ημγ.$

66. Νὰ ἀποδειχθῇ ὅτι : $α^2εφ^2θ+β^2σφ^2θ>2αβ$, ἐκτὸς ἐὰν $αεφ^2θ=β$.

Ἀπόδειξις. Ἔχομεν:

$$α^2εφ^2θ+β^2σφ^2θ=(αεφθ-βσφθ)^2+2αβ$$

ἐξ οὗ:

$$α^2εφ^2θ+β^2σφ^2θ>2αβ,$$

ἐκτὸς ἐὰν $αεφθ-βσφθ=0$ ἢ $αεφ^2θ=β.$

67. Νὰ ἀποδειχθῇ ὅτι : $1+ημ^2α+ημ^2β>ημα+ημβ+ημαημβ.$

Ἀπόδειξις. Ἔχομεν :

$$\left.\begin{array}{lll}(1-\eta\mu\alpha)^2>0 & \Rightarrow & 1+\eta\mu^2\alpha>2\eta\mu\alpha \\ (1-\eta\mu\beta)^2>0 & \Rightarrow & 1+\eta\mu^2\beta>2\eta\mu\beta \\ (\eta\mu\alpha-\eta\mu\beta)^2>0 & \Rightarrow & \eta\mu^2\alpha+\eta\mu^2\beta>2\eta\mu\alpha\eta\mu\beta\end{array}\right\}$$ Διὰ προσθέσεως τούτων κατὰ μέλη, λαμβάνομεν :

$$1+\eta\mu^2\alpha+\eta\mu^2\beta>\eta\mu\alpha+\eta\mu\beta+\eta\mu\alpha\eta\mu\beta.$$

68. Ἐὰν $\alpha+\beta+\gamma=0$, νὰ ἀποδειχθῇ ὅτι :

$$\Sigma\,\sigma\varphi(\gamma+\alpha-\beta)\sigma\varphi(\alpha+\beta-\gamma)=1.$$

Ἀπόδειξις. Θέτομεν :

$$\left.\begin{array}{l}\beta+\gamma-\alpha=x \\ \gamma+\alpha-\beta=y \\ \alpha+\beta-\gamma=\omega\end{array}\right\} \Rightarrow x+y+\omega=\alpha+\beta+\gamma=0.$$ Ἄρα $x+y=-\omega$,

καὶ $\sigma\varphi(x+y)=\sigma\varphi(-\omega)=-\sigma\varphi\omega$ ἢ $\dfrac{\sigma\varphi x\sigma\varphi y-1}{\sigma\varphi x+\sigma\varphi y}=-\sigma\varphi\omega$

ἢ $\sigma\varphi x\sigma\varphi y+\sigma\varphi x\sigma\varphi\omega+\sigma\varphi y\sigma\varphi\omega=1$ ἢ

$$\sigma\varphi(\beta+\gamma-\alpha)\sigma\varphi(\gamma+\alpha-\beta)+\sigma\varphi(\beta+\gamma-\alpha)\sigma\varphi(\alpha+\beta-\gamma)+\sigma\varphi(\gamma+\alpha-\beta)\sigma\varphi(\alpha+\beta-\gamma)=1$$

ἢ $\Sigma\,\sigma\varphi(\gamma+\alpha-\beta)\sigma\varphi(\alpha+\beta-\gamma)=1.$

69. Νὰ ἀποδειχθῇ ὅτι : $\Sigma\,\sigma\varphi(2\alpha+\beta-3\gamma)\sigma\varphi(2\beta+\gamma-3\alpha)=1.$

Ἀπόδειξις. Θέτομεν $2\alpha+\beta-3\gamma=x$, $2\beta+\gamma-3\alpha=y$, $2\gamma+\alpha-3\beta=\omega$,

ὁπότε $x+y+\omega=0$ ἢ $x+y=-\omega$ ἢ $\sigma\varphi(x+y)=\sigma\varphi(-\omega)=-\sigma\varphi\omega$

ἢ $\dfrac{\sigma\varphi x\sigma\varphi y-1}{\sigma\varphi x+\sigma\varphi y}=-\sigma\varphi\omega$ ἢ $\sigma\varphi x\sigma\varphi y+\sigma\varphi y\sigma\varphi\omega+\sigma\varphi\omega\sigma\varphi x=1$

ἢ $\Sigma\,\sigma\varphi(2\alpha+\beta-3\gamma)\sigma\varphi(2\beta+\gamma-3\alpha)=1.$

70. Ἐὰν $xy+y\omega+\omega x=1$, νὰ ἀποδειχθῇ ὅτι :

$$\Sigma\,x(1-y^2)(1-\omega^2)=4xy\omega.$$

Ἀπόδειξις. Θέτομεν $x=\sigma\varphi\alpha$, $y=\sigma\varphi\beta$, $\omega=\sigma\varphi\gamma$, ὅτε

$$\sigma\varphi\beta\sigma\varphi\gamma+\sigma\varphi\gamma\sigma\varphi\alpha+\sigma\varphi\alpha\sigma\varphi\beta=1$$

ἢ $\sigma\varphi\alpha=-\dfrac{\sigma\varphi\beta\sigma\varphi\gamma-1}{\sigma\varphi\gamma+\sigma\varphi\beta}=-\sigma\varphi(\beta+\gamma)$

ἐξ οὗ : $\alpha=k\pi-(\beta+\gamma)$ ἢ $\alpha+\beta+\gamma=k\pi.$

ἢ $2\alpha+2\beta+2\gamma=2k\pi.$

ἐξ οὗ : $\sigma\varphi2\beta\sigma\varphi2\gamma+\sigma\varphi2\gamma\sigma\varphi2\alpha+\sigma\varphi2\alpha\sigma\varphi2\beta=1$

ἢ $\dfrac{(y^2-1)(\omega^2-1)}{4y\omega}+\dfrac{(\omega^2-1)(x^2-1)}{4\omega x}+\dfrac{(x^2-1)(y^2-1)}{4xy}=1$

ἢ $\Sigma\,x(1-y^2)(1-\omega^2)=4xy\omega.$

71. Νὰ ἀποδειχθῇ ὅτι :

$$(2\sigma\upsilon\nu\theta-1)(2\sigma\upsilon\nu2\theta-1)(2\sigma\upsilon\nu4\theta-1)\cdots(2\sigma\upsilon\nu2^{\nu-1}\theta-1)=\frac{2\sigma\upsilon\nu2^{\nu}\,\theta+1}{2\sigma\upsilon\nu\theta+1}.$$

Ἀπόδειξις. Ἔχομεν :

$$(2\sigma\upsilon\nu\theta+1)(2\sigma\upsilon\nu\theta-1)=4\sigma\upsilon\nu^2\theta-1=2(1+\sigma\upsilon\nu2\theta)-1=2\sigma\upsilon\nu2\theta+1.$$

Ὁμοίως : $(2\sigma\upsilon\nu2\theta+1)(2\sigma\upsilon\nu2\theta-1)=2\sigma\upsilon\nu2^2\theta+1$

: $(2\sigma\upsilon\nu2^2\theta+1)(2\sigma\upsilon\nu2^2\theta-1)=2\sigma\upsilon\nu2^3\theta+1,$

: . . . .

: $(2\sigma\upsilon\nu2^{\nu-1}\theta+1)(2\sigma\upsilon\nu2^{\nu-1}\theta-1)=2\sigma\upsilon\nu2^{\nu}\,\theta+1.$

Διὰ πολ/σμοῦ τούτων κατὰ μέλη, λαμβάνομεν :

$$(2\sigma\upsilon\nu\theta-1)(2\sigma\upsilon\nu2\theta-1)(2\sigma\upsilon\nu4\theta-1)\ldots(2\sigma\upsilon\nu2^{\nu-1}\theta-1)=\frac{2\sigma\upsilon\nu2^{\nu}\,\theta+1}{2\sigma\upsilon\nu\theta+1}.$$

ΚΕΦΑΛΑΙΟΝ ΙΙ.

ΤΡΟΠΗ ΑΘΡΟΙΣΜΑΤΩΝ ΕΙΣ ΓΙΝΟΜΕΝΟΝ ΠΑΡΑΓΟΝΤΩΝ

72. Νὰ γίνουν γινόμενα αἱ παραστάσεις:

1. **ημ4α+ημα,**

Λύσις. $ημ4α+ημα = 2ημ\frac{5α}{2}\ συν\frac{3α}{2}$

2. **ημ7α—ημ5α.**

Λύσις. $ημ7α-ημ5α=2ημασυν6α.$

3. **ημ70°+ημ50°.**

Λύσις. $ημ70^{0}+ημ50^{0}=2ημ60^{0}συν10^{0}=\sqrt{3}\,συν10.$

4. **συν3α+συν7α.**

Λύσις. $συν3α+συν7α=2συν5ασυν(-2α)=2συν5ασυν2α.$

5. **ημ2α—ημ4α.**

Λύσις. $ημ2α-ημ4α=2ημ(-α)συν3α=-2ημασυν3α.$

6. **συν5α—συνα.**

Λύσις. $συν5α-συνα=2ημ3αημ(-2α)=-2ημ3αημ2α.$

7. **συν3α—συν5α.**

Λύσις. $συν3α-συν5α=2ημ4ημα,$

8. **συν10°—συν50°.**

Λύσις. $συν10^{0}-συν50^{0}=2ημ30^{0}ημ20^{0}=2\cdot\frac{1}{2}\cdot ημ20^{0}=ημ20^{0}$

73. Νὰ ἀποδειχθῇ ὅτι:

1· $$\frac{συν3α-συν5α}{ημ5α-ημ3α}=εφ4α.$$

Λύσις. $$\frac{συν3α-συν5α}{ημ5α-ημ3α}=\frac{2ημ4αημα}{2ημασυν4α}=\frac{ημ4α}{συν4α}=εφ4α.$$

Τὸ α΄ μέλος ἔχει ἔννοιαν διὰ $α\neq kπ$ καὶ $α\neq k_1\frac{π}{4}+\frac{π}{8}$, $(k, k_1)\in \mathbf{Z}$.

2. $$\frac{συν2α-συν4α}{ημ4α-ημ2α}=εφ3α.$$

Λύσις. Τὸ α΄ μέλος ἔχει ἔννοιαν διὰ $α\neq kπ$ καὶ $α\neq k_1\frac{π}{3}+\frac{π}{6}$, $(k,k_1)\in \mathbf{Z}$.

$$\frac{συν2α-συν4α}{ημ4α-ημ2α}=\frac{2ημ3αημα}{2ημασυν3α}=\frac{ημ3α}{συν3α}=εφ3α.$$

3. $$\frac{\eta\mu 2\alpha+\eta\mu 3\alpha}{\sigma\upsilon\nu 2\alpha-\sigma\upsilon\nu 3\alpha}=\sigma\varphi\frac{\alpha}{2}.$$

Τὸ α΄ μέλος ἔχει ἔννοιαν διὰ $\alpha\neq 2k\cdot\frac{\pi}{5}$ καὶ $\alpha\neq 2k_1\pi$, $(k,k_1)\in\mathbf{Z}$

Λύσις. $$\frac{\eta\mu 2\alpha+\eta\mu 3\alpha}{\sigma\upsilon\nu 2\alpha-\sigma\upsilon\nu 3\alpha}=\frac{2\eta\mu\frac{5\alpha}{2}\sigma\upsilon\nu\frac{\alpha}{2}}{2\eta\mu\frac{5\alpha}{2}\eta\mu\frac{\alpha}{2}}=\frac{\sigma\upsilon\nu\frac{\alpha}{2}}{\eta\mu\frac{\alpha}{2}}=\sigma\varphi\frac{\alpha}{2}.$$

4. $$\frac{\sigma\upsilon\nu 4\alpha-\sigma\upsilon\nu\alpha}{\eta\mu\alpha-\eta\mu 4\alpha}=\varepsilon\varphi\frac{5\alpha}{2}.$$

Τὸ α΄ μέλος ἔχει ἔννοιαν διὰ $\alpha\neq -2k\cdot\frac{\pi}{3}$ καὶ $\alpha\neq 2k_1\frac{\pi}{5}+\frac{\pi}{5}$, $(k,k_1)\in\mathbf{Z}$.

Λύσις. $$\frac{\sigma\upsilon\nu 4\alpha-\sigma\upsilon\nu\alpha}{\eta\mu\alpha-\eta\mu 4\alpha}=\frac{2\eta\mu\frac{5\alpha}{2}\eta\mu\left(-\frac{3\alpha}{2}\right)}{2\eta\mu\left(-\frac{3\alpha}{2}\right)\sigma\upsilon\nu\frac{5\alpha}{2}}=\frac{\eta\mu\frac{5\alpha}{2}}{\sigma\upsilon\nu\frac{5\alpha}{2}}=\varepsilon\varphi\frac{5\alpha}{2},$$

74. Νὰ γίνουν γινόμενα παραγόντων αἱ παραστάσεις:

1. $$\eta\mu\alpha-\eta\mu 2\alpha+\eta\mu 3\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\alpha-\eta\mu 2\alpha+\eta\mu 3\alpha=(\eta\mu 3\alpha+\eta\mu\alpha)-\eta\mu 2\alpha=2\eta\mu 2\alpha\sigma\upsilon\nu\alpha-\eta\mu 2\alpha=$$
$$=2\eta\mu 2\alpha\left(\sigma\upsilon\nu\alpha-\frac{1}{2}\right)=2\eta\mu 2\alpha(\sigma\upsilon\nu\alpha-\sigma\upsilon\nu 60^0)=$$
$$=2\eta\mu 2\alpha\cdot 2\eta\mu\left(30^0+\frac{\alpha}{2}\right)\eta\mu\left(30^0-\frac{\alpha}{2}\right)$$
$$=4\eta\mu 2\alpha\eta\mu\left(30^0+\frac{\alpha}{2}\right)\eta\mu\left(30^0-\frac{\alpha}{2}\right).$$

2. $$\eta\mu 3\alpha+\eta\mu 7\alpha+\eta\mu 10\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu 3\alpha+\eta\mu 7\alpha+\eta\mu 10\alpha=(\eta\mu 7\alpha+\eta\mu 3\alpha)+\eta\mu 10\alpha=$$
$$=2\eta\mu 5\alpha\sigma\upsilon\nu 2\alpha+2\eta\mu 5\alpha\sigma\upsilon\nu 5\alpha=2\eta\mu 5\alpha(\sigma\upsilon\nu 2\alpha+\sigma\upsilon\nu 5\alpha)=$$
$$=2\eta\mu 5\alpha\cdot 2\sigma\upsilon\nu\frac{7\alpha}{2}\sigma\upsilon\nu\frac{3\alpha}{2}=4\eta\mu 5\alpha\sigma\upsilon\nu\frac{7\alpha}{2}\sigma\upsilon\nu\frac{3\alpha}{2}.$$

3. $$\eta\mu\alpha+2\eta\mu 2\alpha+\eta\mu 3\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\alpha+2\eta\mu 2\alpha+\eta\mu 3\alpha=(\eta\mu 3\alpha+\eta\mu\alpha)+2\eta\mu 2\alpha=$$
$$=2\eta\mu 2\alpha\sigma\upsilon\nu\alpha+2\eta\mu 2\alpha=2\eta\mu 2\alpha(1+\sigma\upsilon\nu\alpha)=$$
$$=2\eta\mu 2\alpha\cdot 2\sigma\upsilon\nu^2\frac{\alpha}{2}=4\eta\mu 2\alpha\sigma\upsilon\nu^2\frac{\alpha}{2}.$$

4. $$\mathbf{συνα+2συν2α+συν3α.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συνα+2συν2α+συν3α=(συν3α+συνα)+2συν2α=2συν2ασυνα+2συν2α=$$
$$=2συν2α(1+συνα)=2συν2α\cdot 2συν^2\frac{2α}{2}=4συν2ασυν^2\frac{2α}{2}.$$

5. $$\mathbf{συν7α-συν5α+συν3α-συνα.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν7α-συν5α+συν3α-συνα=(συν7α-συν5α)+(συν3α-συνα)=$$
$$=-2ημ6αημα+(-2ημ2αημα)=-2ημα(ημ6α+ημ2α)=$$
$$=-2ημα\cdot 2ημ4ασυν2α=-4ημα\cdot ημ4α\cdot συν2α.$$

6. $$\mathbf{ημ7α-ημ5α-ημ3α+ημα.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ7α-ημ5α-ημ3α+ημα=(ημ7α-ημ5α)-(ημ3α-ημα)=$$
$$=2ημασυν6α-2ημασυν2α=2ημα(συν6α-συν2α)=$$
$$=2ημα\cdot 2ημ4αημ(-2α)=-4ημαημ2αημ4α.$$

7. $$\mathbf{συν3α+συν5α+συν7α+συν15α.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν3α+συν5α+συν7α+συν15α=(συν7α+συν3α)+(συν15α+συν5α)=$$
$$=2συν5ασυν2α+2συν10ασυν5α=2συν5α(συν2α+συν10α)=$$
$$=2συν5α\cdot 2συν6ασυν4α=4συν4ασυν5ασυν6α.$$

8. $$\mathbf{ημ^2 5α-ημ^2 3α.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ^2 5α-ημ^2 3α=(ημ5α+ημ3α)(ημ5α-ημ3α)=$$
$$=2ημ4ασυνα\cdot 2ημα\cdot συν4α=2ημασυνα\cdot 2ημ4ασυν4α=ημ2α\cdot ημ8α.$$

75. Νὰ ἀποδειχθῇ ὅτι:

1. $$\frac{\mathbf{ημ2α+ημ5α-ημα}}{\mathbf{συν2α+συν5α+συνα}}=\mathbf{εφ2α.}$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $α\neq k\frac{π}{2}+\frac{π}{4}$, $α\neq 2k_1\cdot\frac{π}{3}\pm\frac{2π}{9}$, $(k,k_1)\in \mathbf{Z}$

$$\frac{ημ2α+ημ5α-ημα}{συν2α+συν5α+συνα}=\frac{ημ2α+2ημ2ασυν3α}{συν2α+2συν3ασυν2α}=$$
$$=\frac{ημ2α(2συν3α+1)}{συν2α(2συν3α+1)}=\frac{ημ2α}{συν2α}=εφ2α.$$

2. $$\frac{\eta\mu\alpha+\mu\cdot\eta\mu3\alpha+\eta\mu5\alpha}{\eta\mu3\alpha+\mu\cdot\eta\mu5\alpha+\eta\mu7\alpha}=\frac{\eta\mu3\alpha}{\eta\mu5\alpha}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\alpha\neq k\frac{\pi}{5}$.

$$\frac{\eta\mu\alpha+\mu\cdot\eta\mu3\alpha+\eta\mu5\alpha}{\eta\mu3\alpha+\mu\cdot\eta\mu5\alpha+\eta\mu7\alpha}=\frac{(\eta\mu5\alpha+\eta\mu\alpha)+\mu\cdot\eta\mu3\alpha}{(\eta\mu7\alpha+\eta\mu3\alpha)+\mu\cdot\eta\mu5\alpha}=$$

$$=\frac{2\eta\mu3\alpha\sigma\upsilon\nu2\alpha+\mu\eta\mu3\alpha}{2\eta\mu5\alpha\sigma\upsilon\nu2\alpha+\mu\eta\mu5\alpha}=\frac{\eta\mu3\alpha(2\sigma\upsilon\nu2\alpha+\mu)}{\eta\mu5\alpha(2\sigma\upsilon\nu2\alpha+\mu)}=\frac{\eta\mu3\alpha}{\eta\mu5\alpha}.$$

3. $$\frac{\sigma\upsilon\nu6\alpha+6\sigma\upsilon\nu4\alpha+15\sigma\upsilon\nu2\alpha+10}{\sigma\upsilon\nu5\alpha+5\sigma\upsilon\nu3\alpha+10\sigma\upsilon\nu\alpha}=2\sigma\upsilon\nu\alpha.$$

Λύσις. Ὁ ἀριθμητὴς γράφεται:

$$\sigma\upsilon\nu6\alpha+6\sigma\upsilon\nu4\alpha+15\sigma\upsilon\nu2\alpha+10=(32\sigma\upsilon\nu^6\alpha-48\sigma\upsilon\nu^4\alpha+18\sigma\upsilon\nu^2\alpha-1)+$$
$$+6(8\sigma\upsilon\nu^4\alpha-8\sigma\upsilon\nu^2\alpha+1)+15(2\sigma\upsilon\nu^2\alpha-1)+10=32\sigma\upsilon\nu^6\alpha.$$

Ὁ παρονομαστὴς γράφεται:

$$\sigma\upsilon\nu5\alpha+5\sigma\upsilon\nu3\alpha+10\sigma\upsilon\nu\alpha=(16\sigma\upsilon\nu^5\alpha-20\sigma\upsilon\nu^3\alpha+5\sigma\upsilon\nu\alpha)+5(4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha)+$$
$$+10\sigma\upsilon\nu\alpha=16\sigma\upsilon\nu^5\alpha.$$

Ἄρα τὸ κλάσμα γράφεται: $\frac{32\sigma\upsilon\nu^6\alpha}{16\sigma\upsilon\nu^5\alpha}=2\sigma\upsilon\nu\alpha$,

ἂν $$\alpha\neq k\frac{\pi}{5}+\frac{\pi}{10},\quad k\in\mathbf{Z}.$$

4. $$\frac{\eta\mu\alpha+\eta\mu3\alpha+\eta\mu5\alpha+\eta\mu7\alpha}{\sigma\upsilon\nu\alpha+\sigma\upsilon\nu3\alpha+\sigma\upsilon\nu5\alpha+\sigma\upsilon\nu7\alpha}=\varepsilon\varphi4\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{(\eta\mu3\alpha+\eta\mu\alpha)+(\eta\mu7\alpha+\eta\mu5\alpha)}{(\sigma\upsilon\nu3\alpha+\sigma\upsilon\nu\alpha)+(\sigma\upsilon\nu7\alpha+\sigma\upsilon\nu5\alpha)}=\frac{2\eta\mu2\alpha\sigma\upsilon\nu\alpha+2\eta\mu6\alpha\sigma\upsilon\nu\alpha}{2\sigma\upsilon\nu2\alpha\sigma\upsilon\nu\alpha+2\sigma\upsilon\nu6\alpha\sigma\upsilon\nu\alpha}=$$

$$=\frac{2\sigma\upsilon\nu\alpha(\eta\mu2\alpha+\eta\mu6\alpha)}{2\sigma\upsilon\nu\alpha(\sigma\upsilon\nu2\alpha+\sigma\upsilon\nu6\alpha)}=\frac{\eta\mu6\alpha+\eta\mu2\alpha}{\sigma\upsilon\nu6\alpha+\sigma\upsilon\nu2\alpha}=\frac{2\eta\mu4\alpha\sigma\upsilon\nu2\alpha}{2\sigma\upsilon\nu4\alpha\sigma\upsilon\nu2\alpha}=$$

$$=\frac{\eta\mu4\alpha}{\sigma\upsilon\nu4\alpha}=\varepsilon\varphi4\alpha.$$

ἂν $$\alpha\neq k\pi+\frac{\pi}{2},\quad \alpha\neq k_1\frac{\pi}{4}+\frac{\pi}{8},\quad k,k_1\in\mathbf{Z}.$$

5. $$\frac{\eta\mu(\alpha-\gamma)+2\eta\mu\alpha+\eta\mu(\alpha+\gamma)}{\eta\mu(\beta-\gamma)+2\eta\mu\beta+\eta\mu(\beta+\gamma)}=\frac{\eta\mu\alpha}{\eta\mu\beta}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\eta\mu(\alpha-\gamma)+\eta\mu(\alpha+\gamma)+2\eta\mu\alpha}{\eta\mu(\beta-\gamma)+\eta\mu(\beta+\gamma)+2\eta\mu\beta}=\frac{2\eta\mu\alpha\sigma\upsilon\nu\gamma+2\eta\mu\alpha}{2\eta\mu\beta\sigma\upsilon\nu\gamma+2\eta\mu\beta}=\frac{2\eta\mu\alpha(\sigma\upsilon\nu\gamma+1)}{2\eta\mu\beta(\sigma\upsilon\nu\gamma+1)}=\frac{\eta\mu\alpha}{\eta\mu\beta}$$

ἂν $\beta\neq k\pi$ καὶ $\gamma\neq2k_1\pi\pm\pi$, $(k,k_1)\in\mathbf{Z}$·

6. $$\frac{\eta\mu\alpha+\eta\mu2\alpha+\eta\mu4\alpha+\eta\mu5\alpha}{\sigma\upsilon\nu\alpha+\sigma\upsilon\nu2\alpha+\sigma\upsilon\nu4\alpha+\sigma\upsilon\nu5\alpha}=\varepsilon\varphi3\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{(\eta\mu5\alpha+\eta\mu\alpha)+(\eta\mu4\alpha+\eta\mu2\alpha)}{(\sigma\upsilon\nu5\alpha+\sigma\upsilon\nu\alpha)+(\sigma\upsilon\nu4\alpha+\sigma\upsilon\nu2\alpha)}=\frac{2\eta\mu3\alpha\sigma\upsilon\nu2\alpha+2\eta\mu3\alpha\sigma\upsilon\nu\alpha}{2\sigma\upsilon\nu3\alpha\sigma\upsilon\nu2\alpha+2\sigma\upsilon\nu3\alpha\sigma\upsilon\nu\alpha}=$$

$$=\frac{2\eta\mu3\alpha(\sigma\upsilon\nu2\alpha+\sigma\upsilon\nu\alpha)}{2\sigma\upsilon\nu3\alpha(\sigma\upsilon\nu2\alpha+\sigma\upsilon\nu\alpha)}=\frac{\eta\mu3\alpha}{\sigma\upsilon\nu3\alpha}=\varepsilon\varphi3\alpha.$$

ἂν $\alpha\neq k\cdot\frac{\pi}{3}+\frac{\pi}{6},\quad \alpha\neq2k_1\frac{\pi}{3}+\frac{\pi}{3},\quad \alpha\neq2k_2\pi+\pi,\quad (k,k_1,k_2)\in\mathbf{Z}.$

7. $$\frac{\sigma\upsilon\nu7\alpha+\sigma\upsilon\nu3\alpha-\sigma\upsilon\nu5\alpha-\sigma\upsilon\nu\alpha}{\eta\mu7\alpha-\eta\mu3\alpha-\eta\mu5\alpha+\eta\mu\alpha}=\sigma\varphi2\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{(\sigma\upsilon\nu7\alpha+\sigma\upsilon\nu3\alpha)-(\sigma\upsilon\nu5\alpha+\sigma\upsilon\nu\alpha)}{(\eta\mu7\alpha-\eta\mu3\alpha)-(\eta\mu5\alpha-\eta\mu\alpha)}=\frac{2\sigma\upsilon\nu5\alpha\sigma\upsilon\nu2\alpha-2\sigma\upsilon\nu3\alpha\sigma\upsilon\nu2\alpha}{2\eta\mu2\alpha\sigma\upsilon\nu5\alpha-2\eta\mu2\alpha\sigma\upsilon\nu3\alpha}=$$

$$=\frac{2\sigma\upsilon\nu2\alpha(\sigma\upsilon\nu5\alpha-\sigma\upsilon\nu3\alpha)}{2\eta\mu2\alpha(\sigma\upsilon\nu5\alpha-\sigma\upsilon\nu3\alpha)}=\frac{\sigma\upsilon\nu2\alpha}{\eta\mu2\alpha}=\sigma\varphi2\alpha,$$

ἂν $\alpha\neq k\cdot\frac{\pi}{4},\quad \alpha\neq k_1\frac{\pi}{4},\quad \alpha\neq k_2\pi,\quad (k,k_1,k_2)\in\mathbf{Z}.$

76. Νὰ γίνουν γινόμενα αἱ παραστάσεις :

1. $\mathbf{A=\eta\mu(\alpha+\beta+\gamma)+\eta\mu(\alpha-\beta-\gamma)+\eta\mu(\alpha+\beta-\gamma)+\eta\mu(\alpha-\beta+\gamma)}.$

Λύσις. Ἔχομεν :

$$A=2\eta\mu\frac{\alpha+\beta+\gamma+\alpha-\beta-\gamma}{2}\sigma\upsilon\nu\frac{\alpha+\beta+\gamma-\alpha+\beta+\gamma}{2}+$$

$$+2\eta\mu\frac{\alpha+\beta-\gamma+\alpha-\beta+\gamma}{2}\sigma\upsilon\nu\frac{\alpha+\beta-\gamma-\alpha+\beta-\gamma}{2}=$$

$$=2\eta\mu\alpha\sigma\upsilon\nu(\beta+\gamma)+2\eta\mu\alpha\sigma\upsilon\nu(\beta-\gamma)=2\eta\mu\alpha[\sigma\upsilon\nu(\beta+\gamma)+\sigma\upsilon\nu(\beta-\gamma)]=$$

$$=2\eta\mu\alpha\cdot2\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma=4\eta\mu\alpha\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma.$$

2. $\mathbf{B=\sigma\upsilon\nu(\beta+\gamma-\alpha)-\sigma\upsilon\nu(\gamma+\alpha-\beta)+\sigma\upsilon\nu(\alpha+\beta-\gamma)-\sigma\upsilon\nu(\alpha+\beta+\gamma)}.$

Λύσις. Ἔχομεν :

$$B=2\eta\mu\frac{\beta+\gamma-\alpha+\gamma+\alpha-\beta}{2}\eta\mu\frac{\gamma+\alpha-\beta-\beta-\gamma+\alpha}{2}+$$

$$+2\eta\mu\frac{\alpha+\beta-\gamma+\alpha+\beta+\gamma}{2}\eta\mu\frac{\alpha+\beta+\gamma-\alpha-\beta+\gamma}{2}=$$

$$=2\eta\mu\gamma\eta\mu(\alpha-\beta)+2\eta\mu(\alpha+\beta)\eta\mu\gamma=2\eta\mu\gamma[\eta\mu(\alpha-\beta)+\eta\mu(\alpha+\beta)]=$$

$$=2\eta\mu\gamma\cdot2\eta\mu\frac{\alpha-\beta+\alpha+\beta}{2}\sigma\upsilon\nu\frac{\alpha-\beta-\alpha-\beta}{2}=4\eta\mu\alpha\sigma\upsilon\nu\beta\eta\mu\gamma.$$

3. $\boldsymbol{\Gamma=\eta\mu(\alpha+\beta-\gamma)+\eta\mu(\beta+\gamma-\alpha)+\eta\mu(\gamma+\alpha-\beta)-\eta\mu(\alpha+\beta+\gamma)}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\Gamma=2\eta\mu\frac{\alpha+\beta-\gamma+\beta+\gamma-\alpha}{2}\sigma\upsilon\nu\frac{\alpha+\beta-\gamma-\beta-\gamma+\alpha}{2}+$$

$$+2\eta\mu\frac{\gamma+\alpha-\beta-\alpha-\beta-\gamma}{2}\sigma\upsilon\nu\frac{\gamma+\alpha-\beta+\alpha+\beta+\gamma}{2}=$$

$$=2\eta\mu\beta\sigma\upsilon\nu(\alpha-\gamma)-2\eta\mu\beta\sigma\upsilon\nu(\alpha+\gamma)=2\eta\mu\beta[\sigma\upsilon\nu(\alpha-\gamma)-\sigma\upsilon\nu(\alpha+\gamma)]=$$

$$=2\eta\mu\beta\cdot 2\eta\mu\frac{\alpha-\gamma+\alpha+\gamma}{2}\eta\mu\frac{\alpha+\gamma-\alpha+\gamma}{2}=4\eta\mu\alpha\eta\mu\beta\eta\mu\gamma.$$

4. $\boldsymbol{\Delta=\eta\mu 2\alpha+\eta\mu 2\beta+\eta\mu 2\gamma-\eta\mu 2(\alpha+\beta+\gamma)}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\Delta=2\eta\mu(\alpha+\beta)\sigma\upsilon\nu(\alpha-\beta)+2\eta\mu\frac{2\gamma-2\alpha-2\beta-2\gamma}{2}\sigma\upsilon\nu\frac{2\gamma+2\alpha+2\beta+2\gamma}{2}=$$

$$=2\eta\mu(\alpha+\beta)\sigma\upsilon\nu(\alpha-\beta)-2\eta\mu(\alpha+\beta)\sigma\upsilon\nu(\alpha+\beta+2\gamma)=$$

$$=2\eta\mu(\alpha+\beta)[\sigma\upsilon\nu(\alpha-\beta)=\sigma\upsilon\nu(\alpha+\beta+2\gamma)]=$$

$$=2\eta\mu(\alpha+\beta)\,2\eta\mu\frac{\alpha-\beta+\alpha+\beta+2\gamma}{2}\eta\mu\frac{\alpha+\beta+2\gamma-\alpha+\beta}{2}=$$

$$=4\eta\mu(\alpha+\beta)\eta\mu(\beta+\gamma)\eta\mu(\gamma+\alpha).$$

5. **Νὰ ἀποδειχθῇ ὅτι:**

$$\boldsymbol{\eta\mu\alpha+\eta\mu\beta+\eta\mu(\alpha+\beta)=4\sigma\upsilon\nu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\beta}{2}\eta\mu\frac{\alpha+\beta}{2}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\alpha+\eta\mu\beta+\eta\mu(\alpha+\beta)=2\eta\mu\frac{\alpha+\beta}{2}\sigma\upsilon\nu\frac{\alpha-\beta}{2}+2\eta\mu\frac{\alpha+\beta}{2}\sigma\upsilon\nu\frac{\alpha+\beta}{2}=$$

$$=2\eta\mu\frac{\alpha+\beta}{2}\cdot\left[\sigma\upsilon\nu\frac{\alpha-\beta}{2}+\sigma\upsilon\nu\frac{\alpha+\beta}{2}\right]=2\eta\mu\frac{\alpha+\beta}{2}\cdot 2\sigma\upsilon\nu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\beta}{2}=$$

$$=4\sigma\upsilon\nu\frac{\alpha}{2}\sigma\upsilon\nu\frac{\beta}{2}\eta\mu\frac{\alpha+\beta}{2}.$$

6. $$\boldsymbol{\eta\mu\alpha+\eta\mu\beta-\eta\mu(\alpha+\beta)=4\eta\mu\frac{\alpha}{2}\eta\mu\frac{\beta}{2}\eta\mu\frac{\alpha+\beta}{2}}$$

Λύσις. Ἔχομεν: $\eta\mu\alpha+\eta\mu\beta-\eta\mu(\alpha+\beta)=$

$$=2\eta\mu\frac{\alpha+\beta}{2}\sigma\upsilon\nu\frac{\alpha-\beta}{2}-2\eta\mu\frac{\alpha+\beta}{2}\sigma\upsilon\nu\frac{\alpha+\beta}{2}=$$

$$=2\eta\mu\frac{\alpha+\beta}{2}\left[\sigma\upsilon\nu\frac{\alpha-\beta}{2}-\sigma\upsilon\nu\frac{\alpha+\beta}{2}\right]=$$

$$=2\eta\mu\frac{\alpha+\beta}{2}\cdot 2\eta\mu\frac{\alpha}{2}\eta\mu\frac{\beta}{2}=4\eta\mu\frac{\alpha}{2}\eta\mu\frac{\beta}{2}\eta\mu\frac{\alpha+\beta}{2}.$$

77. Νὰ ἁπλοποιηθοῦν τὰ κλάσματα :

1. $$A = \frac{ημ3α+συν3α+ημ5α+συν5α+ημ7α+συν7α}{συν3α+συν5α+συν7α}.$$

Λύσις. Ὁ ἀριθμητὴς γράφεται :

$$(ημ3α+ημ5α+ημ7α)+(συν3α+συν5α+συν7α) =$$
$$= [(ημ7α+ημ3α)+ημ5α]+[(συν7α+συν3α)+συν5α] =$$
$$= 2ημ5ασυν2α+ημ5α+3συν5ασυν2α+συν5α=$$
$$= ημ5α(2συν2α+1)+συν5α(2συν2α+1)=(2συν2α+1)(ημ5α+συν5α) =$$
$$= (2συν2α+1)\cdot\sqrt{2}\,συν(45^0-5α).$$

Ὁ παρονομαστὴς γράφεται :

$$συν7α+συν3α+συν5α=2συν5ασυν2α+συν5α = συν5α(2συν2α+1).$$

Ἄρα τὸ κλάσμα γράφεται :

$$A = \frac{(2συν2α+1)\cdot\sqrt{2}\,συν(45^0-5α)}{συν5α\cdot(2συν2α+1)} = \frac{\sqrt{2}\,συν(45^0-5α)}{συν5α},$$

ἂν $α \neq k\cdot\frac{π}{5}+\frac{π}{10},\quad α \neq k_1π \pm \frac{π}{3},\quad (k, k_1) \in \mathbf{Z}.$

2. $$B = \frac{συν(α+β+γ)+συν(β+γ-α)+συν(γ+α-β)+συν(α+β-γ)}{ημ(α+β+γ)+ημ(β+γ-α)-ημ(γ+α-β)+ημ(α+β-γ)}.$$

Λύσις. Ὁ ἀριθμητὴς γράφεται :

$$[συν(α+β+γ)+συν(β+γ-α)]+[συν(γ+α-β)+συν(α+β-γ)]=$$
$$= 2συν\frac{α+β+γ+β+γ-α}{2}\,συν\frac{α+β+γ-β-γ+α}{2}+$$
$$+ 2συν\frac{γ+α-β+α+β-γ}{2}\,συν\frac{γ+α-β-α-β+γ}{2}=$$
$$= 2συν(β+γ)συνα+2συνασυν(γ-β) = 2συνα[συν(β+γ)+συν(γ-β)] =$$
$$= 2συνα\cdot 2συνγσυνβ=4συνασυνβσυνγ.$$

Ὁ παρονομαστὴς γράφεται :

$$[ημ(α+β+γ)+ημ(β+γ-α)]-[ημ(γ+α-β)-ημ(α+β-γ)]=$$
$$=2ημ\frac{α+β+γ+β+γ-α}{2}\,συν\frac{α+β+γ-β-γ+α}{2}-$$
$$-2ημ\frac{γ+α-β-α-β+γ}{2}\,συν\frac{γ+α-β+α+β-γ}{2}=$$
$$=2ημ(β+γ)συνα-2ημ(γ-β)συνα=2συνα[ημ(β+γ)-ημ(γ-β)]=$$
$$= 2συνα\cdot 2ημ\frac{β+γ-γ+β}{2}\,συν\frac{β+γ+γ-β}{2}=4συναημβσυνγ.$$

Ἄρα τὸ κλάσμα γράφεται :

$$B = \frac{4συνασυνβσυνγ}{4συναημβσυνγ} = σφβ.$$

ἂν $α \neq kπ+\frac{π}{2},\quad β \neq k_1π,\quad γ \neq k_2π+\frac{π}{2},\quad (k, k_1, k_2) \in \mathbf{Z}.$

72. Νὰ γίνουν γινόμενον παραγόντων αἱ παραστάσεις:

1. $$\mathbf{A=\eta\mu^2 x+\eta\mu^2 y-\eta\mu^2(x-y).}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$A=\frac{1-\sigma\upsilon\nu 2x}{2}+\frac{1-\sigma\upsilon\nu 2y}{2}-\frac{1-\sigma\upsilon\nu(2x-2y)}{2}=$$

$$=\frac{1}{2}-\frac{1}{2}\left[\sigma\upsilon\nu 2x+\sigma\upsilon\nu 2y-\sigma\upsilon\nu(2x-2y)\right]=$$

$$=\frac{1}{2}-\frac{1}{2}\left[2\,\sigma\upsilon\nu(x+y)\sigma\upsilon\nu(x-y)-2\sigma\upsilon\nu^2(x-y)+1\right]=$$

$$=\frac{1}{2}-\sigma\upsilon\nu(x-y)\left[\sigma\upsilon\nu(x+y)-\sigma\upsilon\nu(x-y)\right]-\frac{1}{2}=$$

$$=-\sigma\upsilon\nu(x-y)\left[2\eta\mu\frac{x+y+x-y}{2}\eta\mu\frac{x-y-x-y}{2}\right]=$$

$$=-2\sigma\upsilon\nu(x-y)\eta\mu x\,\eta\mu(-y)=2\eta\mu x\,\eta\mu y\,\sigma\upsilon\nu(x-y).$$

2. $$\mathbf{B=\sigma\upsilon\nu^2(x+y)+\sigma\upsilon\nu^2(x-y)-1.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$B=\frac{1+\sigma\upsilon\nu(2x+2y)}{2}+\frac{1+\sigma\upsilon\nu(2x-2y)}{2}-1=\frac{1}{2}+\frac{1}{2}\sigma\upsilon\nu(2x+2y)+$$

$$+\frac{1}{2}+\frac{1}{2}\sigma\upsilon\nu(2x-2y)-1=\frac{1}{2}\left[\sigma\upsilon\nu(2x+2y)+\sigma\upsilon\nu(2x-2y)\right]=$$

$$=\frac{1}{2}\cdot 2\sigma\upsilon\nu\frac{2x+2y+2x-2y}{2}\sigma\upsilon\nu\frac{2x+2y-2x+2y}{2}=\sigma\upsilon\nu 2x\,\sigma\upsilon\nu 2y.$$

3. $$\mathbf{\Gamma=\sigma\upsilon\nu^2\theta+\sigma\upsilon\nu^2 2\theta+\sigma\upsilon\nu^2 3\theta+\sigma\upsilon\nu^2 4\theta-2.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\Gamma=\frac{1+\sigma\upsilon\nu 2\theta}{2}+\frac{1+\sigma\upsilon\nu 4\theta}{2}+\frac{1+\sigma\upsilon\nu 6\theta}{2}+\frac{1+\sigma\upsilon\nu 8\theta}{2}-2=$$

$$=\frac{1}{2}\left[\sigma\upsilon\nu 2\theta+\sigma\upsilon\nu 4\theta+\sigma\upsilon\nu 6\theta+\sigma\upsilon\nu 8\theta\right]=$$

$$=\frac{1}{2}\left[2\sigma\upsilon\nu 3\theta\sigma\upsilon\nu\theta+2\sigma\upsilon\nu 7\theta\sigma\upsilon\nu\theta\right]=\sigma\upsilon\nu\theta(\sigma\upsilon\nu 3\theta+\sigma\upsilon\nu 7\theta)=$$

$$=\sigma\upsilon\nu\theta\cdot 2\sigma\upsilon\nu 5\theta\sigma\upsilon\nu 2\theta=2\sigma\upsilon\nu\theta\sigma\upsilon\nu 2\theta\sigma\upsilon\nu 5\theta.$$

4. $$\mathbf{\Delta=\eta\mu^2\theta+\eta\mu^2 2\theta+\eta\mu^2 3\theta+\eta\mu^2 4\theta-2.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\Delta=\frac{1-\sigma\upsilon\nu 2\theta}{2}+\frac{1-\sigma\upsilon\nu 4\theta}{2}+\frac{1-\sigma\upsilon\nu 6\theta}{2}+\frac{1-\sigma\upsilon\nu 8\theta}{2}-2=$$

$$=-\frac{1}{2}(\sigma\upsilon\nu 2\theta+\sigma\upsilon\nu 4\theta+\sigma\upsilon\nu 6\theta+\sigma\upsilon\nu 8\theta)=-2\sigma\upsilon\nu\theta\sigma\upsilon\nu 2\theta\sigma\upsilon\nu 5\theta.$$

5. $\mathbf{E=συν^2θ+συν^2 2θ+συν^2 3θ+συν^2 4θ+συν^2 5θ+συν^2 6-3}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$E=\frac{1+συν2θ}{2}+\frac{1+συν4θ}{2}+\frac{1+συν6θ}{2}+\frac{1+συν8θ}{2}+\frac{1+συν10θ}{2}+$$

$$+\frac{1+συν12θ}{2}-3=\frac{1}{2}[συν2θ+συν4θ+συν6θ+συν8θ+συν10θ+συν12θ]=$$

$$=\frac{1}{2}\cdot[2συν3θσυνθ+2συν7θσυνθ+2συν11θσυνθ]=$$

$$=συνθ[συν3θ+συν7θ+συν11θ]=συνθ[2συν7θσυν4θ+συν7θ]=$$

$$=συνθ\cdot συν7θ[2συν4θ+1]=2συνθσυν7θ\left[συν4θ+\frac{1}{2}\right]=$$

$$=2συνθσυν7θ\cdot[συν4θ+συν60^\circ]=2συνθσυν7θ\cdot 2συν(2θ+30^\circ)συν(2θ-30^\circ)=$$

$$=4συνθσυν7θσυν(2θ+30^\circ)συν(2θ-30^\circ).$$

6. $\mathbf{Z=ημ^2θ+ημ^2 2θ+ημ^2 3θ+ημ^2 4θ+ημ^2 5θ+ημ^2 6θ-3}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$Z=\frac{1-συν2θ}{2}+\frac{1-συν4θ}{2}+\frac{1-συν6θ}{2}+\frac{1-συν8θ}{2}+\frac{1-συν10θ}{2}+$$

$$+\frac{1-συν12θ}{2}-3=-\frac{1}{2}[συν2θ+συν4θ+συν6θ+συν8θ+συν10θ+συν12θ]=$$

$$=-4συνθσυν7θσυν(2θ+30^\circ)συν(2θ-30^\circ).$$

79. Νὰ ἀποδειχθῇ ὅτι ἡ παράστασις:

$E=1+ημα+συνα+ημασυνα$ εἶναι τέλειον τετράγωνον.

Λύσις. Ἔχομεν διαδοχικῶς:

$$E=(1+συνα)+(ημα+ημασυνα)=(1+συνα)+ημα(1+συνα)=$$

$$=(1+συνα)(1+ημα)=2συν^2\frac{α}{2}\cdot 2συν^2\left(45^\circ-\frac{α}{2}\right)=$$

$$=\left[2συν\frac{α}{2}συν\left(45^\circ-\frac{α}{2}\right)\right]^2.$$

80. Νὰ ἀποδειχθῇ ὅτι:

1. $$\frac{ημA-ημB}{συνA+συνB}=εφ\frac{A-B}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{ημA-ημB}{συνA+συνB}=\frac{2ημ\frac{A-B}{2}συν\frac{A+B}{2}}{2συν\frac{A+B}{2}συν\frac{A-B}{2}}=\frac{ημ\frac{A-B}{2}}{συν\frac{A-B}{2}}=εφ\frac{A-B}{2}$$

ἂν $\frac{A+B}{2} \neq k\pi + \frac{\pi}{2}$ ἢ $A+B \neq 2k\pi+\pi$, $k \in \mathbf{Z}$

καὶ $\frac{A-B}{2} \neq k_1\pi + \frac{\pi}{2}$ ἢ $A-B \neq 2k_1\pi+\pi$, $k_1 \in \mathbf{Z}$.

2. $$\frac{\eta\mu A+\eta\mu B}{\eta\mu A-\eta\mu B} = \frac{\varepsilon\varphi\frac{A+B}{2}}{\varepsilon\varphi\frac{A-B}{2}}.$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α' μέλος, δέον $\eta\mu A-\eta\mu B \neq 0$

ἢ $$2\eta\mu\frac{A-B}{2}\,\sigma\upsilon\nu\frac{A+B}{2} \neq 0,$$ ὅθεν

$$\frac{A-B}{2} \neq k\pi \quad \text{ἢ} \quad A-B \neq 2k\pi, \quad k \in \mathbf{Z}$$

καὶ $\frac{A+B}{2} \neq k_1\pi + \frac{\pi}{2}$ ἢ $A+B \neq 2k_1\pi+\pi$. $k_1 \in \mathbf{Z}$.

Ἑπομένως θὰ ἔχωμεν:

$$\frac{\eta\mu A+\eta\mu B}{\eta\mu A-\eta\mu B} = \frac{2\eta\mu\frac{A+B}{2}\,\sigma\upsilon\nu\frac{A-B}{2}}{2\eta\mu\frac{A-B}{2}\,\sigma\upsilon\nu\frac{A+B}{2}} = \frac{\varepsilon\varphi\frac{A+B}{2}}{\varepsilon\varphi\frac{A-B}{2}}$$

ἂν $\frac{A-B}{2} \neq k_3\pi + \frac{\pi}{2}$ ἢ $A-B \neq 2k_3\pi+\pi$, $k_3 \in \mathbf{Z}$.

3. $$\frac{\sigma\upsilon\nu A-\sigma\upsilon\nu B}{\sigma\upsilon\nu A+\sigma\upsilon\nu B} = \frac{\varepsilon\varphi\frac{B+A}{2}}{\varepsilon\varphi\frac{B-A}{2}}.$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α' μέλος, πρέπει:

$\sigma\upsilon\nu A+\sigma\upsilon\nu B \neq 0$ ἢ $2\sigma\upsilon\nu\frac{A+B}{2}\,\sigma\upsilon\nu\frac{A-B}{2} \neq 0$,

ἐξ οὗ $\frac{A+B}{2} \neq k\pi + \frac{\pi}{2}$ ἢ $A+B \neq 2k\pi+\pi$, $k \in \mathbf{Z}$

καὶ $\frac{A-B}{2} \neq k_1\pi + \frac{\pi}{2}$ ἢ $A-B \neq 2k_1\pi+\pi$, $k_1 \in \mathbf{Z}$.

Ἑπομένως θὰ ἔχωμεν διαδοχικῶς:

$$\frac{\sigma\upsilon\nu A-\sigma\upsilon\nu B}{\sigma\upsilon\nu A+\sigma\upsilon\nu B} = \frac{2\eta\mu\frac{A+B}{2}\,\eta\mu\frac{B-A}{2}}{2\sigma\upsilon\nu\frac{A+B}{2}\,\sigma\upsilon\nu\frac{A-B}{2}} = \frac{\varepsilon\varphi\frac{A+B}{2}}{\sigma\varphi\frac{B-A}{2}} = \frac{\varepsilon\varphi\frac{B+A}{2}}{\sigma\varphi\frac{B-A}{2}},$$

ἂν $\frac{B-A}{2} \neq k_3\pi + \frac{\pi}{2}$ ἢ $B-A \neq 2k_3\pi+\pi$, $z_3 \in \mathbf{Z}$.

4. $$\frac{\sigma\upsilon\nu A+\sigma\upsilon\nu B}{\sigma\upsilon\nu B-\sigma\upsilon\nu A}=\frac{\sigma\varphi\frac{A+B}{2}}{\sigma\varphi\frac{A-B}{2}}.$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α΄ μέλος, πρέπει :

$$\sigma\upsilon\nu B-\sigma\upsilon\nu A\neq 0 \quad ἢ \quad 2\eta\mu\frac{B+A}{2}\,\eta\mu\frac{A-B}{2}\neq 0,$$

ὅθεν $$\frac{A+B}{2}\neq k\pi, \quad ἐξ\ οὗ \quad A+B\neq 2k\pi, \quad k\in \mathbf{Z}$$

$$\frac{A-B}{2}\neq k_1\pi, \quad ἐξ\ οὗ \quad A-B\neq 2k_1\pi, \quad k_1\in \mathbf{Z}.$$

Κατ' ἀκολουθίαν θὰ ἔχωμεν:

$$\frac{\sigma\upsilon\nu A+\sigma\upsilon\nu B}{\sigma\upsilon\nu B-\sigma\upsilon\nu A}=\frac{2\sigma\upsilon\nu\frac{A+B}{2}\,\sigma\upsilon\nu\frac{A-B}{2}}{2\eta\mu\frac{A+B}{2}\,\eta\mu\frac{A-B}{2}}=\frac{\sigma\varphi\frac{A+B}{2}}{\varepsilon\varphi\frac{A-B}{2}}$$

ἂν $$\frac{A-B}{2}\neq k_2\pi+\frac{\pi}{2} \quad ἢ \quad A-B\neq 2k_2\pi+\pi, \quad k_2\in \mathbf{Z}.$$

ΜΕΤΑΣΧΗΜΑΤΙΣΜΟΣ ΓΙΝΟΜΕΝΩΝ ΕΙΣ ΑΘΡΟΙΣΜΑ ἢ ΔΙΑΦΟΡΑΝ

81. Νὰ μετασχηματισθοῦν εἰς ἄρθροισμα ἢ διαφορὰν αἱ παραστάσεις:

1. **2ημ2ασυνα.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\eta\mu2\alpha\sigma\upsilon\nu\alpha=\eta\mu(2\alpha+\alpha)+\eta\mu(2\alpha-\alpha)=\eta\mu3\alpha+\eta\mu\alpha.$$

2. **2συν2ασυνα.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\sigma\upsilon\nu2\alpha\sigma\upsilon\nu\alpha=\sigma\upsilon\nu(2\alpha+\alpha)+\sigma\upsilon\nu(2\alpha-\alpha)=\sigma\upsilon\nu3\alpha+\sigma\upsilon\nu\alpha.$$

3. **3ημασυν4α.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\eta\mu\alpha\sigma\upsilon\nu4\alpha=\eta\mu(\alpha+4\alpha)+\eta\mu(\alpha-4\alpha)=\eta\mu5\alpha-\eta\mu3\alpha.$$

4. **2ημαημ3α.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\eta\mu\alpha\eta\mu3\alpha=\sigma\upsilon\nu(\alpha-3\alpha)-\sigma\upsilon\nu(\alpha+3\alpha)=\sigma\upsilon\nu2\alpha-\sigma\upsilon\nu4\alpha.$$

5. **2ημ4ασυν8α.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\eta\mu4\alpha\sigma\upsilon\nu8\alpha=\eta\mu(4\alpha+8\alpha)+\eta\mu(4\alpha-8\alpha)=\eta\mu12\alpha-\eta\mu4\alpha.$$

6. **$2\sigma\upsilon\nu 5\alpha\sigma\upsilon\nu 7\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\sigma\upsilon\nu 5\alpha\sigma\upsilon\nu 7\alpha = \sigma\upsilon\nu(5\alpha+7\alpha)+\sigma\upsilon\nu(5\alpha-7\alpha)=\sigma\upsilon\nu 12\alpha+\sigma\upsilon\nu 2\alpha.$$

7. **$2\eta\mu 5\alpha\eta\mu 3\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\eta\mu 5\alpha\eta\mu 3\alpha=\sigma\upsilon\nu(5\alpha-3\alpha)-\sigma\upsilon\nu(5\alpha+3\alpha)=\sigma\upsilon\nu 2\alpha-\sigma\upsilon\nu 8\alpha.$$

8. **$2\eta\mu 3\alpha\eta\mu 5\alpha$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\eta\mu 3\alpha\eta\mu 5\alpha=\sigma\upsilon\nu(3\alpha-5\alpha)-\sigma\upsilon\nu(3\alpha+5\alpha)=\sigma\upsilon\nu 2\alpha-\sigma\upsilon\nu 8\alpha.$$

82. Νὰ εὑρεθῇ ἡ ἀριθμητικὴ τιμὴ τῶν παραστάσεων:

1. **$2\sigma\upsilon\nu 60^{\circ}\eta\mu 30^{\circ}$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\sigma\upsilon\nu 60^{\circ}\eta\mu 30^{\circ}=\eta\mu(30^{\circ}+60^{\circ})+\eta\mu(30^{\circ}-60^{\circ})=\eta\mu 90^{\circ}-\eta\mu 30^{\circ}=1-\frac{1}{2}=\frac{1}{2}.$$

2. **$2\sigma\upsilon\nu 45^{\circ}\sigma\upsilon\nu 63^{\circ}$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\sigma\upsilon\nu 45^{\circ}\sigma\upsilon\nu 63^{\circ}=\sigma\upsilon\nu(45^{\circ}+63^{\circ})+\sigma\upsilon\nu(45^{\circ}-63^{\circ})=\sigma\upsilon\nu 108^{\circ}+\sigma\upsilon\nu 18^{\circ}=$$

$$=\sigma\upsilon\nu(90^{\circ}+18^{\circ})+\sigma\upsilon\nu 18^{\circ}=-\eta\mu 18^{\circ}+\sigma\upsilon\nu 18^{\circ}=$$

$$=-\frac{\sqrt{5}-1}{4}+\frac{\sqrt{10+2\sqrt{5}}}{4}=\frac{\sqrt{10+2\sqrt{5}}-\sqrt{5}+1}{4}.$$

3. **$\eta\mu 45^{\circ}\sigma\upsilon\nu 75^{\circ}$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu 45^{\circ}\cdot\sigma\upsilon\nu 75^{\circ}=\frac{1}{2}[\eta\mu(45^{\circ}+75^{\circ})+\eta\mu(45^{\circ}-75^{\circ})]=\frac{1}{9}[\eta\mu 120^{\circ}-\eta\mu 30^{\circ}]=$$

$$=\frac{1}{2}\left[\frac{\sqrt{3}}{2}-\frac{1}{2}\right]=\frac{\sqrt{3}-1}{4}.$$

4. **$2\sigma\upsilon\nu 150^{\circ}\sigma\upsilon\nu 30^{\circ}$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\sigma\upsilon\nu 150^{\circ}\sigma\upsilon\nu 30^{\circ}=\sigma\upsilon\nu(150^{\circ}+30^{\circ})+\sigma\upsilon\nu(150^{\circ}-30^{\circ})=$$

$$=\sigma\upsilon\nu 180^{\circ}+\sigma\upsilon\nu 120^{\circ}=-1-\frac{1}{2}=-\frac{3}{2}.$$

5. **$\eta\mu 75^{\circ}\eta\mu 30^{\circ}$.**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu 75^{\circ}\eta\mu 30^{\circ}=\frac{1}{2}[\sigma\upsilon\nu(75^{\circ}-30^{\circ})-\sigma\upsilon\nu(75^{\circ}+30^{\circ})]=\frac{1}{2}[\sigma\upsilon\nu 45^{\circ}-\sigma\upsilon\nu 105^{\circ}]=$$

$$= \frac{1}{2}\left[\text{συν}45^0 - \text{συν}(90^0+15^0)\right] = \frac{1}{2}\left[\ \text{συν}45^0 + \text{ημ}15^0\ \right] =$$

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{6}-\sqrt{2}}{4}\right] = \frac{2\sqrt{2}}{8} + \frac{\sqrt{6}-\sqrt{2}}{8} = \frac{\sqrt{6}+\sqrt{2}}{8}.$$

6. **2ημ60°συν45°.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$2\text{ημ}60^0\text{συν}45^0 = \text{ημ}(60^0+45^0) + \text{ημ}(60^0-45^0) = \text{ημ}105^0 + \text{ημ}15^0 =$$

$$= \text{ημ}(90^0+15^0) + \text{ημ}15^0 = \text{συν}15^0 + \text{ημ}15^0 =$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4} + \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{2\sqrt{6}}{4} = \frac{\ \ }{2}.$$

7. **συν42°συν48°.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{συν}42^0\text{συν}48^0 = \frac{1}{2}\left[\text{συν}(42^0+48^0) + \text{συν}(42^0-48^0)\right] = \frac{1}{2}\left[\text{συν}90^0 + \text{συν}6^0\right] =$$

$$= \frac{1}{2}\left[0 + \text{συν}6^0 = \frac{1}{2}\ \text{συν}6^0 = \frac{1}{2}\cdot\left[\frac{1}{8}\sqrt{10-2\sqrt{5}} + \frac{\sqrt{3}}{8}(1+\sqrt{5})\right] =\right.$$

$$= \frac{1}{16}\left[\sqrt{10-2\sqrt{5}} + \sqrt{3}(1+\sqrt{5})\right].$$

Σημ. Ἐπειδὴ $3^0 = 48^0 - 45^0$, ἕπεται ὅτι :

$$\text{ημ}3^0 = \text{ημ}(48^0-45^0)\ldots \text{ Ἄρα } \text{συν}3^0 = \sqrt{1-\text{ημ}^2 3^0} = \ldots$$

καὶ $$\text{συν}6^0 = 1 - 2\text{ημ}^2 3^0 = 2\text{συν}^2 3^0 - 1 = \ldots$$

8. **2ημ36°συν54°.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$2\text{ημ}36^0\text{συν}54^0 = \text{ημ}(36^0+54^0) + \text{ημ}(36^0-54^0) = \text{ημ}90^0 - \text{ημ}18^0 =$$

$$= 1 - \frac{\sqrt{5}-1}{4} = \frac{4-\sqrt{5}+1}{4} = \frac{5-\sqrt{5}}{4}.$$

83. Νὰ ἀποδειχθῇ ὅτι :

1. **συν2ασυνα—ημ4αημα=συν3ασυν2α.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{συν}2\alpha\text{συν}\alpha - \text{ημ}4\alpha\text{ημ}\alpha = \frac{1}{2}\left[\text{συν}3\alpha + \text{συν}\alpha - \text{συν}3\alpha + \text{συν}5\alpha\right] =$$

$$= \frac{1}{2}\left[\text{συν}\alpha + \text{συν}5\alpha\right] = \frac{1}{2}\cdot 2\text{συν}3\alpha\text{συν}2\alpha = \text{συν}3\alpha\text{συν}5\alpha.$$

2. $\textbf{συν5ασυν2α—συν4ασυν3α=—ημ2αημα.}$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν5ασυν2α-συν4ασυν3α=\frac{1}{2}[συν7α+συν3α-συν7α-συνα]=$$

$$=\frac{1}{2}[συν3α-συνα]=\frac{1}{2}[2ημ2αημ(-α)]=-ημ2αημα.$$

3. $\textbf{ημ4ασυνα—ημ3ασυν2α=ημασυν2α.}$

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ4ασυνα-ημ3ασυν2α=\frac{1}{2}[ημ(4α+α)+ημ(4α-α)]-\frac{1}{2}[ημ(3α+2α)+ημ(3α-2α)]$$

$$=\frac{1}{2}(ημ5α+ημ3α-ημ5α-ημα)=$$

$$=\frac{1}{2}(ημ3α-ημα)=ημασυν2α.$$

4. $$\boldsymbol{ημ\frac{α}{2}ημ\frac{7α}{2}+ημ\frac{3α}{2}ημ\frac{11α}{2}=ημ2αημ5α.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$ημ\frac{α}{2}ημ\frac{7α}{2}+ημ\frac{3α}{2}ημ\frac{11α}{2}=\frac{1}{2}\left[συν\left(\frac{α}{2}-\frac{7α}{2}\right)-συν\left(\frac{α}{2}+\frac{7α}{2}\right)\right]+$$

$$+\frac{1}{2}\left[συν\left(\frac{3α}{2}-\frac{11α}{2}\right)-συν\left(\frac{3α}{2}+\frac{11α}{2}\right)\right]=$$

$$=\frac{1}{2}[συν3α-συν4α+συν4α-συν7α]=\frac{1}{2}(συν3α-συν7α)=$$

$$=\frac{1}{2}\cdot 2ημ5αημ2α=ημ2αημ5α.$$

5. $$\boldsymbol{συν2ασυν\frac{α}{2}-συν3ασυν\frac{9α}{2}=ημ5α\cdot ημ\frac{5α}{2}.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν2ασυν\frac{α}{2}-συν3ασυν\frac{9α}{2}=\frac{1}{2}\left(συν\frac{5α}{2}+συν\frac{3α}{2}\right)-\frac{1}{2}\left(συν\frac{15α}{2}+συν\frac{3α}{2}\right)$$

$$=\frac{1}{2}\left(συν\frac{5α}{2}-συν\frac{15α}{2}\right)=$$

$$=\frac{1}{2}\cdot 2ημ5α\cdot ημ\frac{5α}{2}=ημ5α\cdot ημ\frac{5α}{2}.$$

84. Νὰ ἀποδειχθῇ ὅτι:

1. $\textbf{συν(36°—α)συν(36°+α)+συν(54°+α)συν(54°—α)=σύν2α.}$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν(36^0-α)συν(36^0+α)+συν(54^0+α)συν(54^0-α)=$$

$$= \frac{1}{2}[\sigma\upsilon\nu(36^0-\alpha+36^0+\alpha)+\sigma\upsilon\nu(36^0-\alpha-36^0-\alpha)] +$$

$$+ \frac{1}{2}[\sigma\upsilon\nu(54^0+\alpha+54^0-\alpha)+\sigma\upsilon\nu(54^0+\alpha-54^0+\alpha)]$$

$$= \frac{1}{2}[\sigma\upsilon\nu72^0+\sigma\upsilon\nu2\alpha] + \frac{1}{2}[\sigma\upsilon\nu108^0+\sigma\upsilon\nu2\alpha] =$$

$$= \frac{1}{2}[\sigma\upsilon\nu72^0+\sigma\upsilon\nu108^0+2\sigma\upsilon\nu2\alpha] =$$

$$= \frac{1}{2}[\sigma\upsilon\nu72^0-\sigma\upsilon\nu72^0+2\sigma\upsilon\nu2\alpha] = \sigma\upsilon\nu2\alpha.$$

2. **$\sigma\upsilon\nu\alpha\eta\mu(\beta-\gamma)+\sigma\upsilon\nu\beta\eta\mu(\gamma-\alpha)+\sigma\upsilon\nu\gamma\eta\mu(\alpha-\beta)=0.$**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu\alpha\eta\mu(\beta-\gamma)= \frac{1}{2}[\eta\mu(\beta-\gamma+\alpha)+\eta\mu(\beta-\gamma-\alpha)]= \frac{1}{2}[\eta\mu(\beta-\gamma+\alpha)-\eta\mu(\alpha+\gamma-\beta)]$$

$$\sigma\upsilon\nu\beta\eta\mu(\gamma-\alpha) = \frac{1}{2}[\eta\mu(\gamma-\alpha+\beta)+\eta\mu(\gamma-\alpha-\beta)]= \frac{1}{2}[\eta\mu(\gamma-\alpha+\beta)-\eta\mu(\beta+\alpha-\gamma)]$$

$$\sigma\upsilon\nu\gamma\eta\mu(\alpha-\beta)= \frac{1}{2}[\eta\mu(\alpha-\beta+\gamma)+\eta\mu(\alpha-\beta-\gamma)]= \frac{1}{2}[\eta\mu(\alpha-\beta+\gamma)-\eta\mu(\beta+\gamma-\alpha)]$$

Ἄρα:

$$\alpha' \ \mu\acute{\varepsilon}\lambda o\varsigma = \frac{1}{2}[\eta\mu(\beta-\gamma+\alpha)-\eta\mu(\alpha+\gamma-\beta)+\eta\mu(\gamma-\alpha+\beta)-\eta\mu(\beta+\alpha-\gamma)+$$

$$+ \eta\mu(\alpha-\beta+\gamma)-\eta\mu(\beta+\gamma-\alpha)] = \frac{1}{2}\cdot 0 = 0.$$

3. **$\eta\mu\alpha\eta\mu(\alpha+2\beta)-\eta\mu\beta\eta\mu(\beta+2\alpha)=\eta\mu(\alpha-\beta)\eta\mu(\alpha+\beta).$**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\alpha\eta\mu(\alpha+2\beta)-\eta\mu\beta\eta\mu(\beta+2\alpha) =$$

$$= \frac{1}{2}[\sigma\upsilon\nu(\alpha-\alpha-2\beta)-\sigma\upsilon\nu(\alpha+\alpha+2\beta)) - \frac{1}{2}[\sigma\upsilon\nu(\beta-\beta-2\alpha)-\sigma\upsilon\nu(\beta+\beta+2\alpha)]$$

$$= \frac{1}{2}[\sigma\upsilon\nu2\beta-\sigma\upsilon\nu(2\alpha+2\beta)-\sigma\upsilon\nu2\alpha+\sigma\upsilon\nu(2\alpha+2\beta)] = \frac{1}{2}[\sigma\upsilon\nu2\beta-\sigma\upsilon\nu2\alpha] =$$

$$= \frac{1}{2}\cdot 2\eta\mu(\beta+\alpha)\eta\mu(\alpha-\beta)=\eta\mu(\alpha-\beta)\eta\mu(\alpha+\beta).$$

4. **$(\eta\mu3\alpha+\eta\mu\alpha)\eta\mu\alpha+(\sigma\upsilon\nu3\alpha-\sigma\upsilon\nu\alpha)\sigma\upsilon\nu\alpha=0.$**

Λύσις. Ἔχομεν διαδοχικῶς:

$$(\eta\mu3\alpha+\eta\mu\alpha)\eta\mu\alpha+(\sigma\upsilon\nu3\alpha-\sigma\upsilon\nu\alpha)\sigma\upsilon\nu\alpha=$$

$$=(3\eta\mu\alpha-4\eta\mu^3\alpha+\eta\mu\alpha)\eta\mu\alpha+(4\sigma\upsilon\nu^3\alpha-3\sigma\upsilon\nu\alpha-\sigma\upsilon\nu\alpha)\sigma\upsilon\nu\alpha=$$

$$=(4\eta\mu\alpha-4\eta\mu^3\alpha)\eta\mu\alpha+(4\sigma\upsilon\nu^3\alpha-4\sigma\upsilon\nu\alpha)\sigma\upsilon\nu\alpha=$$

$$=4\eta\mu^2\alpha(1-\eta\mu^2\alpha)-4\sigma\upsilon\nu^2\alpha(1-\sigma\upsilon\nu^2\alpha)=4\eta\mu^2\alpha\sigma\upsilon\nu^2\alpha-4\sigma\upsilon\nu^2\alpha\eta\mu^2\alpha=0.$$

5. **$\eta\mu\alpha\eta\mu(\beta-\gamma)+\eta\mu\beta\eta\mu(\gamma-\alpha)+\eta\mu\gamma\eta\mu(\alpha-\beta)=0.$**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\alpha\eta\mu(\beta-\gamma)=\frac{1}{2}[\sigma\upsilon\nu(\alpha-\beta+\gamma)-\sigma\upsilon\nu(\alpha+\beta-\gamma)]$$

$$\eta\mu\beta\eta\mu(\gamma-\alpha)=\frac{1}{2}[\sigma\upsilon\nu(\beta-\gamma+\alpha)-\sigma\upsilon\nu(\beta+\gamma-\alpha)]$$

$$\eta\mu\gamma\eta\mu(\alpha-\beta)=\frac{1}{2}[\sigma\upsilon\nu(\gamma-\alpha+\beta)-\sigma\upsilon\nu(\gamma+\alpha-\beta)]$$

Ἄρα $\quad$ α΄ μέλος $=\frac{1}{2}\cdot 0=0.$

6. **$\sigma\upsilon\nu\alpha\eta\mu(\beta-\gamma)+\sigma\upsilon\nu\beta\eta\mu(\gamma-\alpha)+\sigma\upsilon\nu\gamma\eta\mu(\alpha-\beta)=0.$**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu\alpha\eta\mu(\beta-\gamma)=\frac{1}{2}[\eta\mu(\beta-\gamma+\alpha)+\eta\mu(\beta-\gamma-\alpha)]=\frac{1}{2}[\eta\mu(\beta-\gamma+\alpha)-\eta\mu(\gamma+\alpha-\beta)]$$

$$\sigma\upsilon\nu\beta\eta\mu(\gamma-\alpha)=\frac{1}{2}[\eta\mu(\gamma-\alpha+\beta)+\eta\mu(\gamma-\alpha-\beta)]=\frac{1}{2}[\eta\mu(\gamma-\alpha+\beta)-\eta\mu(\alpha+\beta-\gamma)]$$

$$\sigma\upsilon\nu\gamma\eta\mu(\alpha-\beta)=\frac{1}{2}[\eta\mu(\alpha-\beta+\gamma)+\eta\mu(\alpha-\beta-\gamma)]=\frac{1}{2}[\eta\mu(\alpha-\beta+\gamma)-\eta\mu(\beta+\gamma-\alpha)]$$

Ἄρα τὸ πρῶτον μέλος γράφεται: $\quad =\frac{1}{2}\cdot 0=0.$

7. **$\eta\mu(\beta-\gamma)\sigma\upsilon\nu(\alpha-\delta)+\eta\mu(\gamma-\alpha)\sigma\upsilon\nu(\beta-\delta)+\eta\mu(\alpha-\beta)\sigma\upsilon\nu(\gamma-\delta)=0.$**

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu(\beta-\gamma)\sigma\upsilon\nu(\alpha-\delta)=\frac{1}{2}[\eta\mu(\beta-\gamma+\alpha-\delta)+\eta\mu(\beta-\gamma-\alpha+\delta)]=$$

$$=\frac{1}{2}[\eta\mu(\alpha+\beta-\gamma-\delta)-\eta\mu(\alpha+\gamma-\beta-\delta)],$$

$$\eta\mu(\gamma-\alpha)\sigma\upsilon\nu(\beta-\delta)=\frac{1}{2}[\eta\mu(\gamma-\alpha+\beta-\delta)+\eta\mu(\gamma-\alpha-\beta+\delta)]=$$

$$=\frac{1}{2}[\eta\mu(\gamma-\alpha+\beta-\delta)-\eta\mu(\alpha+\beta-\gamma-\delta)],$$

$$\eta\mu(\alpha-\beta)\sigma\upsilon\nu(\gamma-\delta)=\frac{1}{2}[\eta\mu(\alpha-\beta+\gamma-\delta)+\eta\mu(\alpha-\beta-\gamma+\delta)]=$$

$$=\frac{1}{2}[\eta\mu(\alpha+\gamma-\beta-\delta)-\eta\mu(\beta+\gamma-\alpha-\delta)].$$

Ἄρα $\quad$ α΄ μέλος $=\frac{1}{2}\cdot 0=0.$

8. $$\textbf{συν}(\alpha+\beta)\textbf{ημ}(\alpha-\beta)+\textbf{συν}(\beta+\gamma)\textbf{ημ}(\beta-\gamma)+$$
$$+\textbf{συν}(\gamma+\delta)\textbf{ημ}(\gamma-\delta)+\textbf{συν}(\delta+\alpha)\textbf{ημ}(\delta-\alpha)=0.$$

Λύσις. Ἔχομεν:

$$συν(\alpha+\beta)ημ(\alpha-\beta)=\frac{1}{2}[ημ(\alpha+\beta+\alpha-\beta)-ημ(\alpha+\beta-\alpha+\beta)]=\frac{1}{2}(ημ2\alpha-ημ2\beta)$$

$$συν(\beta+\gamma)ημ(\beta-\gamma)=\frac{1}{2}[ημ(\beta+\gamma+\beta-\gamma)-ημ(\beta+\gamma-\beta+\gamma)]=\frac{1}{2}(ημ2\beta-ημ2\gamma)$$

$$συν(\gamma+\delta)ημ(\gamma-\delta)=\frac{1}{2}[ημ(\gamma+\delta+\gamma-\delta)-ημ(\gamma+\delta-\gamma+\delta)]=\frac{1}{2}(ημ2\gamma-ημ2\delta)$$

$$συν(\delta+\alpha)ημ(\delta-\alpha)=\frac{1}{2}[ημ(\delta+\alpha+\delta-\alpha)-ημ(\delta+\alpha-\delta+\alpha)]=\frac{1}{2}(ημ2\delta-ημ2\alpha)$$

Ἄρα α′ μέλος $=\frac{1}{2}\cdot 0=0.$

9. $$\frac{\textbf{ημαημ2α}+\textbf{ημ3αημ6α}+\textbf{ημ4αημ13α}}{\textbf{ημασυν2α}+\textbf{ημ3ασυν6α}+\textbf{ημ4ασυν13α}}=\textbf{εφ9α}.$$

Λύσις. Διὰ νὰ ἔχῃ ἔννοιαν ἀριθμοῦ τὸ α′ μέλος, πρέπει:

$$(ημασυν2α+ημ3ασυν6α+ημ4ασυν13α)\neq 0$$

ἢ $$\frac{1}{2}[ημ3α-ημα+ημ9α-ημ3α+ημ17α-ημ9α]\neq 0$$

ἢ $$(ημ17α-ημα)\neq 0 \quad ἢ \quad 2ημ8ασυν9α\neq 0$$

$$\left.\begin{array}{ll} ἢ & ημ8α\neq 0 \\ καὶ & συν9α\neq 0 \end{array}\right\} \Longrightarrow \left.\begin{array}{l} 8α\neq k\pi \\ 9α\neq k_1\pi+\frac{\pi}{2} \end{array}\right\} \Longrightarrow \begin{array}{ll} α\neq k\frac{\pi}{8}, & k\in \mathbf{Z} \\ α\neq k_1\frac{\pi}{9}+\frac{\pi}{18}, & k_1\in \mathbf{Z} \end{array}$$

Ὁ ἀριθμητὴς γράφεται:

$$ημαημ2α+ημ3αημ6α+ημ4αημ13α=$$
$$=\frac{1}{2}[συνα-συν3α+συν3α-συν9α+συν9α-συν17α]=\frac{1}{2}(συνα-συν17α)=$$
$$=\frac{1}{2}\cdot 2ημ9αημ8α=ημ9αημ8α.$$

Ἄρα τὸ α′ μέλος $=\frac{ημ9α\ ημ8α}{ημ8ασυν9α}=\frac{ημ9α}{συν9α}=εφ9α.$

85. Νὰ ἀποδειχθῇ ὅτι:

1. $$\textbf{συν20°συν40°συν60°συν80°}=\frac{1}{16}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν20^0συν40^0συν60^0συν80^0=\frac{1}{2}συν20^0[συν120^0+συν40^0]συν60^0=$$
$$=\frac{1}{2}συν20^0\left[-\frac{1}{2}+(2συν^2 20^0-1)\right]συν60^0=\frac{1}{2}συν20^0\left(2συν^2 20^0-\frac{3}{2}\right)συν60^0=$$
$$=\frac{1}{4}(4συν^3 20^0-3συν20^0)συν60^0=\frac{1}{4}συν^2 60=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}.$$

2. $$\varepsilon\varphi 20^0 \varepsilon\varphi 40^0 \varepsilon\varphi 60^0 \varepsilon\varphi 80^0 = 3$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\varepsilon\varphi 20^0 \varepsilon\varphi 40^0 \varepsilon\varphi 60^0 \varepsilon\varphi 80^0 = \frac{\eta\mu 20^0 \eta\mu 40^0 \eta\mu 60^0 \eta\mu 80^0}{\sigma\upsilon\nu 20^0 \sigma\upsilon\nu 40^0 \sigma\upsilon\nu 60^0 \sigma\upsilon\nu 80^0} = \frac{\frac{3}{16}}{\frac{1}{16}} = 3.$$

3. $$\sigma\varphi 20^0 \sigma\varphi 40^0 \sigma\varphi 60^0 \sigma\varphi 80^0 = \frac{1}{3}.$$

Λύσις. Ἔχομεν:

$$\sigma\varphi 20^0 \sigma\varphi 40^0 \sigma\varphi 60^0 \sigma\varphi 80^0 = \frac{1}{\varepsilon\varphi 20^0 \varepsilon\varphi 40^0 \varepsilon\varphi 60^0 \varepsilon\varphi 80^0} = \frac{1}{3}.$$

Εἰς τὰς ἀνωτέρω ἀσκήσεις 1, 2, 3 παρατηροῦμεν ὅτι τὰ τόξα 20^0, 40^0, 60^0, 80^0 ἀποτελοῦν ἀριθμητικὴν πρόοδον.

Γενικῶς ἀποδεικνύεται ὅτι: *

4. $$\eta\mu\frac{\pi}{2\nu+1}\cdot\eta\mu\frac{2\pi}{2\nu+1}\eta\mu\frac{3\pi}{2\nu+1}\cdots\eta\mu\frac{\nu\pi}{2\nu+1} = \frac{\sqrt{2\nu+1}}{2^\nu}.$$

5. $$\sigma\upsilon\nu\frac{\pi}{2\nu+1}\,\sigma\upsilon\nu\frac{2\alpha}{2\nu+1}\,\sigma\upsilon\nu\frac{3\pi}{2\nu+1}\cdots\sigma\upsilon\nu\frac{\nu\pi}{2\nu+1} = \frac{1}{2^\nu}.$$

6. $$\varepsilon\varphi\frac{\pi}{2\nu+1}\,\varepsilon\varphi\frac{2\pi}{2\nu+1}\,\varepsilon\varphi\frac{3\pi}{2\nu+1}\cdots\varepsilon\varphi\frac{\nu\pi}{2\nu+1} = \sqrt{2\nu+1}.$$

Ἐνταῦθα παρατηροῦμεν ὅτι τὰ τόξα $\frac{\pi}{2\nu+1}, \frac{2\pi}{2\nu+1}, \ldots, \frac{\nu\pi}{2\nu+1}$ ἀποτελοῦν ἀριθμητικὴν πρόοδον μὲ λόγον $\frac{\pi}{2\nu+1}$. Ἡ ἀπόδειξις γίνεται ὅπως καὶ εἰς τὰς ἀσκήσεις (1), (2), (3).

86. Νὰ ἀποδειχθῇ ὅτι:

1. $$\varepsilon\varphi 6^0 \varepsilon\varphi 42^0 \varepsilon\varphi 66^0 \varepsilon\varphi 78^0 = 1.$$

Λύσις. Ἔχομεν:

$$\varepsilon\varphi 6^0 \varepsilon\varphi 66^0 = \frac{\eta\mu 6^0 \eta\mu 66^0}{\sigma\upsilon\nu 6^0 \sigma\upsilon\nu 66^0} = \frac{\sigma\upsilon\nu 60^0 - \sigma\upsilon\nu 72^0}{\sigma\upsilon\nu 60^0 + \sigma\upsilon\nu 72^0} = \frac{1-2\sigma\upsilon\nu 72^0}{1+2\sigma\upsilon\nu 72^0},$$

$$\varepsilon\varphi 42^0 \varepsilon\varphi 78^0 = \frac{\eta\mu 42^0 \eta\mu 78^0}{\sigma\upsilon\nu 42^0 \sigma\upsilon\nu 78^0} = \frac{\sigma\upsilon\nu 36^0 - \sigma\upsilon\nu 120^0}{\sigma\upsilon\nu 36^0 + \sigma\upsilon\nu 120^0} = \frac{2\sigma\upsilon\nu 36^0 + 1}{2\sigma\upsilon\nu 36^0 - 1}.$$

Ἐπειδὴ δὲ $\sigma\upsilon\nu 72^0 = \frac{\sqrt{5}-1}{4}$ καὶ $\sigma\upsilon\nu 36^0 = \frac{\sqrt{5}+1}{4}$, ἕπεται ὅτι:

$$\varepsilon\varphi 6^0 \varepsilon\varphi 42^0 \varepsilon\varphi 66^0 \varepsilon\varphi 78^0 = \frac{3-\sqrt{5}}{1+\sqrt{5}}\cdot\frac{\sqrt{5}+3}{\sqrt{5}-1} = \frac{9-5}{5-1} = \frac{4}{4} = 1.$$

* Ἡ ἀπόδειξις θὰ γίνῃ εἰς τὸ τέλος τοῦ βιβλίου.

2. $$\textbf{συν}\frac{2\pi}{7}+\textbf{συν}\frac{4\pi}{7}+\textbf{συν}\frac{6\pi}{7}=-\frac{1}{2}.$$

Λύσις. Θέτομεν: $\Sigma=\text{συν}\frac{2\pi}{7}+\text{συν}\frac{4\pi}{7}+\text{συν}\frac{6\pi}{7}$, ὁπότε:

$$\left(2\text{ημ}\frac{\pi}{7}\right)\Sigma=2\text{ημ}\frac{\pi}{7}\text{συν}\frac{2\pi}{7}+2\text{ημ}\frac{\pi}{7}\text{συν}\frac{4\pi}{7}+2\text{ημ}\frac{\pi}{7}\text{συν}\frac{6\pi}{7}=$$

$$=\left(\text{ημ}\frac{3\pi}{7}-\text{ημ}\frac{\pi}{7}\right)+\left(\text{ημ}\frac{5\pi}{7}-\text{ημ}\frac{3\pi}{7}\right)+\left(\text{ημ}\pi-\text{ημ}\frac{5\pi}{7}\right)=-\text{ημ}\frac{\pi}{7},$$

ἐξ οὗ: $$\Sigma=-\frac{1}{2}.$$

3. $$\textbf{2συν}\frac{\pi}{13}\textbf{συν}\frac{9\pi}{13}+\textbf{συν}\frac{3\pi}{13}+\textbf{συν}\frac{5\pi}{13}=0.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$2\text{συν}\frac{\pi}{13}\text{συν}\frac{9\pi}{13}+\text{συν}\frac{3\pi}{13}+\text{συν}\frac{5\pi}{13}=\text{συν}\frac{10\pi}{13}+\text{συν}\frac{8\pi}{13}+\text{συν}\frac{3\pi}{13}+\text{συν}\frac{5\pi}{13}=$$

$$=\text{συν}\frac{10\pi}{13}+\text{συν}\frac{8\pi}{13}-\text{συν}\frac{10\pi}{13}-\text{συν}\frac{8\pi}{13}=0,$$

διότι: $\frac{10\pi}{13}+\frac{3\pi}{13}=\pi$, ἄρα: $\text{συν}\frac{3\pi}{13}=-\text{συν}\frac{10\pi}{13}$

καὶ $\frac{8\pi}{13}+\frac{5\pi}{13}=\pi$, ἄρα: $\text{συν}\frac{5\pi}{13}=-\text{συν}\frac{8\pi}{13}$.

4, $$\textbf{ημ}\frac{\pi}{24}\textbf{ημ}\frac{5\pi}{24}\textbf{ημ}\frac{7\pi}{24}\textbf{ημ}\frac{11\pi}{24}=\frac{1}{16}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\text{ημ}\frac{\pi}{24}\text{ημ}\frac{5\pi}{24}\cdot\text{ημ}\frac{7\pi}{24}\text{ημ}\frac{11\pi}{24}=\frac{1}{2}\left[\text{συν}\frac{\pi}{6}-\text{συν}\frac{\pi}{4}\right]\cdot\frac{1}{2}\left[\text{συν}\frac{\pi}{6}-\text{συν}\frac{3\pi}{4}\right]=$$

$$=\frac{1}{2}\left(\text{συν}\frac{\pi}{6}-\text{συν}\frac{\pi}{4}\right)\cdot\frac{1}{2}\left(\text{συν}\frac{\pi}{6}+\text{συν}\frac{\pi}{4}\right)=$$

$$=\frac{1}{4}\left[\text{συν}^2\frac{\pi}{6}-\text{συν}^2\frac{\pi}{4}\right]=\frac{1}{4}\left(\frac{3}{4}-\frac{2}{4}\right)=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}.$$

5. $$\textbf{εφ}9^0-\textbf{εφ}27^0-\textbf{εφ}63+\textbf{εφ}81^0=4.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\text{εφ}9^0-\text{εφ}27^0-\text{εφ}63^0+\text{εφ}81^0=(\text{εφ}81^0+\text{εφ}9^0)-(\text{εφ}63^0+\text{εφ}27^0)=$$

$$=\frac{\text{ημ}90^0}{\text{συν}81^0\text{συν}9^0}-\frac{\text{ημ}90^0}{\text{συν}63^0\text{συν}27^0}=\frac{2}{\text{συν}90^0+\text{συν}72^0}-\frac{2}{\text{συν}90^0+\text{συν}36^0}=$$

$$=\frac{2(\text{συν}36^0-\text{συν}72^0)}{\text{συν}36^0\text{συν}72^0}=\frac{4\text{ημ}54^0\text{ημ}18^0}{\text{συν}36^0\text{συν}72^0}=4,$$

καθόσον $\text{ημ}54^0=\text{συν}36^0$ καὶ $\text{ημ}18^0=\text{συν}72^0$.

6. $\varepsilon\varphi 36^0 \varepsilon\varphi 72^0 \varepsilon\varphi 108^0 \varepsilon\varphi 144^0 = 5.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\varepsilon\varphi 36^0 \varepsilon\varphi 72^0 \varepsilon\varphi 108^0 \varepsilon\varphi 144^0 = \varepsilon\varphi 36^0 \cdot \varepsilon\varphi 72^0 (-\varepsilon\varphi 72^0)(-\varepsilon\varphi 36^0) = (\varepsilon\varphi 36^0 \varepsilon\varphi 72^0)^2 =$$

$$= \left(\sqrt{5-2\sqrt{5}} \cdot \sqrt{5+2\sqrt{5}}\right)^2 = (5-2\sqrt{5})(5+2\sqrt{5}) = 25-20=5.$$

7. $\eta\mu^4 \frac{\pi}{16} + \eta\mu^4 \frac{3\pi}{16} + \eta\mu^4 \frac{5\pi}{16} + \eta\mu^4 \frac{7\pi}{16} = \frac{3}{2}.$

Λύσις. Ἐπειδὴ $\frac{7\pi}{16} + \frac{\pi}{16} = \frac{8\pi}{16} = \frac{\pi}{2} \Rightarrow \eta\mu^4 \frac{7\pi}{16} = \sigma\upsilon\nu^4 \frac{\pi}{16}$

καὶ ἐπειδὴ $\frac{5\pi}{16} + \frac{3\pi}{16} = \frac{8\pi}{16} = \frac{\pi}{2} \Rightarrow \eta\mu^4 \frac{5\pi}{16} = \sigma\upsilon\nu^4 \frac{3\pi}{16}.$

Ἄρα θὰ ἔχωμεν διαδοχικῶς:

$$\eta\mu^4 \frac{\pi}{16} + \eta\mu^4 \frac{3\pi}{16} + \eta\mu^4 \frac{5\pi}{16} + \eta\mu^4 \frac{7\pi}{16} = \eta\mu^4 \frac{\pi}{16} + \sigma\upsilon\nu^4 \frac{\pi}{16} + \eta\mu^4 \frac{3\pi}{16} + \sigma\upsilon\nu^4 \frac{3\pi}{16} =$$

$$= \left(\eta\mu^2 \frac{\pi}{16}\right)^2 + \left(\sigma\upsilon\nu^2 \frac{\pi}{16}\right)^2 + \left(\eta\mu^2 \frac{3\pi}{16}\right)^2 + \left(\sigma\upsilon\nu^2 \frac{3\pi}{16}\right)^2 =$$

$$= \left(\frac{1-\sigma\upsilon\nu \frac{\pi}{8}}{2}\right)^2 + \left(\frac{1+\sigma\upsilon\nu \frac{\pi}{8}}{2}\right)^2 + \left(\frac{1-\sigma\upsilon\nu \frac{3\pi}{8}}{2}\right)^2 + \left(\frac{1+\sigma\upsilon\nu \frac{3\pi}{8}}{2}\right)^2$$

$$= \left(\frac{1-\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}\right)^2 + \left(\frac{1+\frac{1}{2}\sqrt{2+\sqrt{2}}}{2}\right)^2 + \left(\frac{1-\frac{1}{2}\sqrt{2-\sqrt{2}}}{2}\right)^2 + \left(\frac{1+\frac{1}{2}\sqrt{2-\sqrt{2}}}{2}\right)^2$$

$$= \frac{\left(2-\sqrt{2+\sqrt{2}}\right)^2}{16} + \frac{\left(2+\sqrt{2+\sqrt{2}}\right)^2}{16} + \frac{\left(2-\sqrt{2-\sqrt{2}}\right)^2}{16} + \frac{\left(2+\sqrt{2-\sqrt{2}}\right)^2}{16}$$

$$= \frac{4+2+\sqrt{2}-4\sqrt{2+\sqrt{2}}+2+2+\sqrt{2}+4\sqrt{2+\sqrt{2}}+4+4-\sqrt{2}-4\sqrt{2-\sqrt{2}}}{16} +$$

$$+ \frac{4+2-\sqrt{2}+4\sqrt{2-\sqrt{2}}}{16} = \frac{24}{16} = \frac{3}{2}.$$

87. Νὰ ὑπολογισθοῦν τὰ ἀκόλουθα ἀθροίσματα:

1. $S = \eta\mu 2\alpha + \eta\mu 4\alpha + \eta\mu 6\alpha + \cdots$ **(ἐκ ν ὅρων).**

Λύσις. Ἐὰν εἰς τὸν τύπον (58) θέσωμεν ὅπου α τὸ 2α καὶ ὅπου ω τὸ 2α, εὑρίσκομεν:

$$S = \frac{\eta\mu(\nu\alpha)\sigma\upsilon\nu(\nu+1)\alpha}{\eta\mu\alpha}.$$

2. $S = \sigma\upsilon\nu 2\alpha + \sigma\upsilon\nu 4\alpha + \sigma\upsilon\nu 6\alpha + \cdots$ **(ἐκ ν ὅρων).**

Λύσις. Ἐὰν εἰς τὸν τύπον (59) θέσωμεν ὅπου α τὸ 2α καὶ ὅπου ω τὸ 2α, εὑρίσκομεν:

$$S = \frac{\eta\mu(\nu\alpha)\sigma\upsilon\nu(\nu+1)\alpha}{\eta\mu\alpha}.$$

3. $S = \eta\mu\alpha - \eta\mu 2\alpha + \eta\mu 3\alpha - \eta\mu 4\alpha + \cdots$ **(ἐκ ν ὅρων).**

Λύσις. Τὸ δοθὲν ἄθροισμα γράφεται:

$$S = \eta\mu\alpha + \eta\mu[\alpha + (\pi+\alpha)] + \eta\mu[\alpha + 2(\pi+\alpha)] + \ldots$$

Ἐὰν δὲ εἰς τὸν γενικὸν τύπον (58) θέσωμεν ἀντὶ τοῦ ω τὸ $\pi+\alpha$, λαμβάνομεν:

$$S = \frac{\eta\mu\left[\frac{(\nu+1)\alpha}{2} + \frac{(\nu-1)\pi}{2}\right]\eta\mu\left[\frac{\nu\alpha}{2} + \frac{\nu\pi}{2}\right]}{\sigma\upsilon\nu\frac{\alpha}{2}}$$

4. $S = \sigma\upsilon\nu\alpha - \sigma\upsilon\nu 2\alpha + \sigma\upsilon\nu 3\alpha - \sigma\upsilon\nu 4\alpha + \cdots$ **(ἐκ ν ὅρων).**

Λύσις. Ἐργαζόμενοι ὅπως προηγουμένως, εὑρίσκομεν ὅτι:

$$S = \frac{\sigma\upsilon\nu\left[\frac{(\nu+1)\alpha}{2} + \frac{(\nu-1)\pi}{2}\right]\eta\mu\left[\frac{\nu\alpha}{2} + \frac{\nu\pi}{2}\right]}{\sigma\upsilon\nu\frac{\alpha}{2}}.$$

88. Νὰ ἀποδειχθῇ ὅτι:

1. $\sigma\upsilon\nu\frac{\pi}{19} + \sigma\upsilon\nu\frac{3\pi}{19} + \sigma\upsilon\nu\frac{5\pi}{19} + \cdots + \sigma\upsilon\nu\frac{17\pi}{19} = \frac{1}{2}.$

Λύσις. Τὸ πλῆθος τῶν ὅρων τοῦ πρώτου μέλους εἶναι, προφανῶς, 9 καὶ ὁ λόγος $\omega = \frac{2\pi}{9}$. Ἄρα ὁ γενικὸς τύπος (59) δίδει:

$$S = \frac{\eta\mu\frac{9\pi}{19}}{\eta\mu\frac{\pi}{19}}\cdot\sigma\upsilon\nu\frac{9\pi}{19} = \frac{\eta\mu\frac{18\pi}{19}}{2\eta\mu\frac{\pi}{19}} = \frac{\eta\mu\left(\pi - \frac{\pi}{19}\right)}{2\eta\mu\frac{\pi}{19}} = \frac{\eta\mu\frac{\pi}{19}}{2\eta\mu\frac{\pi}{19}} = \frac{1}{2}.$$

2. $\sigma\upsilon\nu\frac{2\pi}{21} + \sigma\upsilon\nu\frac{4\pi}{21} + \sigma\upsilon\nu\frac{6\pi}{21} + \cdots + \sigma\upsilon\nu\frac{20\pi}{21} = -\frac{1}{2}.$

Λύσις. Ἐνταῦθα ὁ λόγος $\omega = \frac{2\pi}{21}$ καὶ τὸ πλῆθος $\nu = 10$. Ἄρα ὁ τύπος (59) δίδει:

$$S = \frac{\eta\mu\frac{10\pi}{21}}{\eta\mu\frac{\pi}{21}}\sigma\upsilon\nu\frac{11\pi}{21} = \frac{\eta\mu\frac{22\pi}{21}}{2\eta\mu\frac{\pi}{21}} = -\frac{1}{2}.$$

3. $\eta\mu\frac{\pi}{\nu} + \eta\mu\frac{2\pi}{\nu} + \eta\mu\frac{3\pi}{\nu} + \eta\mu\frac{4\pi}{\nu} + \cdots$ **[ἐκ (ν—1) ὅρων]**

Λύσις. Ἐνταῦθα ὁ λόγος $\omega = \frac{\pi}{\nu}$ καὶ τὸ πλῆθος$=\nu-1$. Ἄρα ὁ τύπος (58) δίδει:

$$S=\frac{\eta\mu\frac{(\nu-1)\pi}{2\nu}}{\eta\mu\frac{\pi}{2\nu}}\eta\mu\frac{\pi+(\nu-1)\pi}{2\nu}=\frac{\eta\mu\left(\frac{\pi}{2}-\frac{\pi}{2\nu}\right)}{\eta\mu\frac{\pi}{2\nu}}\eta\mu\frac{\pi}{2}=\sigma\varphi\frac{\pi}{2\nu}.$$

4. $\sigma\upsilon\nu\frac{\pi}{\nu}+\sigma\upsilon\nu\frac{3\pi}{\nu}+\sigma\upsilon\nu\frac{5\pi}{\nu}+\cdots$ **(ἐκ 2ν—1 ὅρων).**

Λύσις. Ἐνταῦθα ὁ λόγος $\omega=\frac{2\pi}{\nu}$ καὶ τὸ πλῆθος $=2\nu-1$.

Ἄρα ὁ τύπος (59) δίδει:

$$S=\frac{\eta\mu\frac{(2\nu-1)\pi}{\nu}}{\eta\mu\frac{\pi}{\nu}}\sigma\upsilon\nu\frac{\pi+[2(2\nu-1)-1]\pi}{2\nu}=\frac{\eta\mu\left(2\pi-\frac{\pi}{\nu}\right)}{\eta\mu\frac{\pi}{\nu}}\sigma\upsilon\nu\frac{(4\nu-2)\pi}{2\nu}=-\sigma\upsilon\nu\frac{\pi}{\nu}.$$

ΤΡΙΓΩΝΟΜΕΤΡΙΚΑΙ ΤΑΥΤΟΤΗΤΕΣ ΑΦΟΡΩΣΑΙ ΕΙΣ ΤΑΣ ΓΩΝΙΑΣ ΤΡΙΓΩΝΟΥ—ΤΕΤΡΑΠΛΕΥΡΟΥ ἢ ΤΑΥΤΟΤΗΤΕΣ ΥΠΟ ΣΥΝΘΗΚΑΣ

89. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $\eta\mu A+\eta\mu B-\eta\mu\Gamma=4\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\sigma\upsilon\nu\frac{\Gamma}{2}.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu A+\eta\mu B-\eta\mu\Gamma=2\eta\mu\frac{A+B}{2}\sigma\upsilon\nu\frac{A-B}{2}-2\eta\mu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{\Gamma}{2}$$
$$=2\sigma\upsilon\nu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{A-B}{2}-2\sigma\upsilon\nu\frac{A+B}{2}\sigma\upsilon\nu\frac{\Gamma}{2}$$
$$=2\sigma\upsilon\nu\frac{\Gamma}{2}\left[\sigma\upsilon\nu\frac{A-B}{2}-\sigma\upsilon\nu\frac{A+B}{2}\right]=4\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\sigma\upsilon\nu\frac{\Gamma}{2}.$$

2. $\sigma\upsilon\nu A+\sigma\upsilon\nu B-\sigma\upsilon\nu\Gamma=-1+4\sigma\upsilon\nu\frac{A}{2}\sigma\upsilon\nu\frac{B}{2}\eta\mu\frac{\Gamma}{2}.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu A+\sigma\upsilon\nu B-\sigma\upsilon\nu\Gamma=2\sigma\upsilon\nu\frac{A+B}{2}\sigma\upsilon\nu\frac{A-B}{2}-1+2\eta\mu^2\frac{\Gamma}{2}$$
$$=2\eta\mu\frac{\Gamma}{2}\left[\sigma\upsilon\nu\frac{A-B}{2}+\sigma\upsilon\nu\frac{A+B}{2}\right]-1$$
$$=-1+4\sigma\upsilon\nu\frac{A}{2}\sigma\upsilon\nu\frac{B}{2}\eta\mu\frac{\Gamma}{2}.$$

3. $\eta\mu 2A+\eta\mu 2B+\eta\mu 2\Gamma=4\eta\mu A\eta\mu B\eta\mu\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\begin{aligned}\eta\mu 2A+\eta\mu 2B+\eta\mu 2\Gamma&=2\eta\mu(A+B)\sigma\upsilon\nu(A-B)+2\eta\mu\Gamma\sigma\upsilon\nu\Gamma\\&=2\eta\mu\Gamma\sigma\upsilon\nu(A-B)-2\eta\mu\Gamma\sigma\upsilon\nu(A+B)\\&=2\eta\mu\Gamma[\sigma\upsilon\nu(A-B)-\sigma\upsilon\nu(A+B)]\\&=4\eta\mu A\eta\mu B\eta\mu\Gamma.\end{aligned}$$

4. $\eta\mu 2A+\eta\mu 2B-\eta\mu 2\Gamma=4\sigma\upsilon\nu A\sigma\upsilon\nu B\eta\mu\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\begin{aligned}\eta\mu 2A+\eta\mu 2B-\eta\mu 2\Gamma&=2\eta\mu(A+B)\sigma\upsilon\nu(A-B)-2\eta\mu\Gamma\sigma\upsilon\nu\Gamma\\&=2\eta\mu\Gamma\sigma\upsilon\nu(A-B)+2\eta\mu\Gamma\sigma\upsilon\nu(A+B)\\&=2\eta\mu\Gamma[\sigma\upsilon\nu(A-B)+\sigma\upsilon\nu(A+B)]\\&=4\sigma\upsilon\nu A\sigma\upsilon\nu B\eta\mu\Gamma.\end{aligned}$$

5. $\sigma\upsilon\nu 2A+\sigma\upsilon\nu 2B+\sigma\upsilon\nu 2\Gamma=-1-4\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma.$

Λύσις. Ἔχομεν διαδοχιχῶς:

$$\begin{aligned}\sigma\upsilon\nu 2A+\sigma\upsilon\nu 2B+\sigma\upsilon\nu 2\Gamma&=2\sigma\upsilon\nu(A+B)\sigma\upsilon\nu(A-B)+2\sigma\upsilon\nu^2\Gamma-1\\&=-1-2\sigma\upsilon\nu\Gamma\sigma\upsilon\nu(A-B)+2\sigma\upsilon\nu^2\Gamma\\&=-1-2\sigma\upsilon\nu\Gamma[\sigma\upsilon\nu(A-B)+\sigma\upsilon\nu(A+B)]\\&=-1-4\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma.\end{aligned}$$

6. $\sigma\upsilon\nu 2A+\sigma\upsilon\nu 2B-\sigma\upsilon\nu 2\Gamma=1-4\eta\mu A\eta\mu B\sigma\upsilon\nu\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\begin{aligned}\sigma\upsilon\nu 2A+\sigma\upsilon\nu 2B-\sigma\upsilon\nu 2\Gamma&=2\sigma\upsilon\nu(A+B)\sigma\upsilon\nu(A-B)+1-2\sigma\upsilon\nu^2\Gamma\\&=1-2\sigma\upsilon\nu\Gamma\sigma\upsilon\nu(A-B)-2\sigma\upsilon\nu^2\Gamma\\&=1-2\sigma\upsilon\nu\Gamma[\sigma\upsilon\nu(A-B)-\sigma\upsilon\nu(A+B)]\\&=1-4\eta\mu A\eta\mu B\sigma\upsilon\nu\Gamma.\end{aligned}$$

7. $\varepsilon\varphi 2A+\varepsilon\varphi 2B+\varepsilon\varphi 2\Gamma=\varepsilon\varphi 2A\varepsilon\varphi 2B\varepsilon\varphi 2\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$A+B+\Gamma=\pi$ ἢ $2A+2B=2\pi-2\Gamma$ ἢ $\varepsilon\varphi(2A+2B)=\varepsilon\varphi(2\pi-2\Gamma)=-\varepsilon\varphi 2\Gamma$

ἢ
$$\frac{\varepsilon\varphi 2A+\varepsilon\varphi 2B}{1-\varepsilon\varphi 2A\varepsilon\varphi 2B}=-\varepsilon\varphi 2\Gamma,$$

ἐξ οὗ:
$$\varepsilon\varphi 2A+\varepsilon\varphi 2B+\varepsilon\varphi 2\Gamma=\varepsilon\varphi 2A\varepsilon\varphi 2B\varepsilon\varphi 2\Gamma.$$

8. **Ἂν** $\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2}+\varepsilon\varphi\frac{B}{2}\,\varepsilon\varphi\frac{\Gamma}{2}+\varepsilon\varphi\frac{\Gamma}{2}\,\varepsilon\varphi\frac{A}{2}=1,$ **τότε**

$$A+B+\Gamma=(2\nu+1)\pi.$$

Λύσις. Ἔχομεν ἐξ ὑποθέσεως ὅτι:

$$\text{εφ}\frac{A}{2}\,\text{εφ}\frac{B}{2}+\text{εφ}\frac{B}{2}\,\text{εφ}\frac{Γ}{2}+\text{εφ}\frac{Γ}{2}\text{εφ}\frac{A}{2}=1 \quad \text{ἢ} \quad \left(\text{εφ}\frac{A}{2}+\text{εφ}\frac{B}{2}\right)\text{εφ}\frac{Γ}{2}=$$

$$=1-\text{εφ}\frac{A}{2}\,\text{εφ}\frac{B}{2} \quad \text{ἢ} \quad \frac{\text{εφ}\frac{A}{2}+\text{εφ}\frac{B}{2}}{1-\text{εφ}\frac{A}{2}\,\text{εφ}\frac{B}{2}}=\frac{1}{\text{εφ}\frac{Γ}{2}}$$

ἢ
$$\text{εφ}\left(\frac{A}{2}+\frac{B}{2}\right)=\text{σφ}\frac{Γ}{2}=\text{εφ}\left(\frac{\pi}{2}-\frac{Γ}{2}\right)$$

ἐξ οὗ: $\frac{A}{2}+\frac{B}{2}=\nu\pi+\frac{\pi}{2}-\frac{Γ}{2}$ ἢ $A+B+Γ=2\nu\pi+\pi=(2\nu+1)\pi$,

ἤτοι
$$\boxed{A+B+Γ=(2\nu+1)\pi}$$

9. **Ἂν** $\text{σφ}\frac{A}{2}+\text{σφ}\frac{B}{2}+\text{σφ}\frac{Γ}{2}=\text{σφ}\frac{A}{2}\,\text{σφ}\frac{B}{2}\,\text{σφ}\frac{Γ}{2}$, **τότε**

$$A+B+Γ=(2\nu+1)\pi.$$

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$\text{σφ}\frac{A}{2}+\text{σφ}\frac{B}{2}=\text{σφ}\frac{A}{2}\,\text{σφ}\frac{B}{2}\,\text{σφ}\frac{Γ}{2}-\text{σφ}\frac{Γ}{2}=\text{σφ}\frac{Γ}{2}\left(\text{σφ}\frac{A}{2}\,\text{σφ}\frac{B}{2}-1\right)$$

ἢ
$$\frac{\text{σφ}\frac{A}{2}\cdot\text{σφ}\frac{B}{2}-1}{\text{σφ}\frac{A}{2}+\text{σφ}\frac{B}{2}}=\frac{1}{\text{σφ}\frac{Γ}{2}} \quad \text{ἢ} \quad \text{σφ}\left(\frac{A}{2}+\frac{B}{2}\right)=\text{εφ}\frac{Γ}{2}=\text{σφ}\left(\frac{\pi}{2}-\frac{Γ}{2}\right)$$

ἢ
$$\frac{A}{2}+\frac{B}{2}=\nu\pi+\frac{\pi}{2}-\frac{Γ}{2} \Longrightarrow \boxed{A+B+Γ=(2\nu+1)\pi}\,.$$

90. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

1. $$\text{ημ}^2A+\text{ημ}^2B+\text{ημ}^2Γ=2+2\text{συν}A\,\text{συν}B\,\text{συν}Γ.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\text{ημ}^2A+\text{ημ}^2B+\text{ημ}^2Γ=\frac{1-\text{συν}2A}{2}+\frac{1-\text{συν}2B}{2}+\frac{1-\text{συν}2Γ}{2}=$$

$$=\frac{3}{2}-\frac{1}{2}(\text{συν}2A+\text{συν}2B+\text{συν}2Γ)=\frac{3}{2}-\frac{1}{2}(-1-4\text{συν}A\,\text{συν}B\,\text{συν}Γ)$$

$$=\frac{3}{2}+\frac{1}{2}+2\text{συν}A\,\text{συν}B\,\text{συν}Γ=2+2\text{συν}A\,\text{συν}B\,\text{συν}Γ.$$

2. $$\text{ημ}^2A+\text{ημ}^2B-\text{ημ}^2Γ=2\text{ημ}A\,\text{ημ}B\,\text{συν}Γ.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu^2 A+\eta\mu^2 B-\eta\mu^2\Gamma=\eta\mu^2 A+\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)=$$
$$=\eta\mu^2 A+\eta\mu A\eta\mu(B-\Gamma)=\eta\mu A[\eta\mu A+\eta\mu(B-\Gamma)]=$$
$$=\eta\mu A[\eta\mu(B+\Gamma)+\eta\mu(B-\Gamma)]=$$
$$=2\eta\mu A\eta\mu B\sigma\upsilon\nu\Gamma.$$

3. $\boldsymbol{\sigma\upsilon\nu^2 A+\sigma\upsilon\nu^2 B-\sigma\upsilon\nu^2\Gamma=1-2\eta\mu A\eta\mu B\sigma\upsilon\nu\Gamma.}$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu^2 A+\sigma\upsilon\nu^2 B-\sigma\upsilon\nu^2\Gamma=\frac{1+\sigma\upsilon\nu 2A}{2}+\frac{1+\sigma\upsilon\nu 2B}{2}-\frac{1+\sigma\upsilon\nu 2\Gamma}{2}=$$
$$=\frac{1}{2}+\frac{1}{2}[\sigma\upsilon\nu 2A+\sigma\upsilon\nu 2B-\sigma\upsilon\nu 2\Gamma]$$
$$=\frac{1}{2}+\frac{1}{2}[1-4\eta\mu A\eta\mu B\sigma\upsilon\nu\Gamma]$$
$$=1-2\eta\mu A\eta\mu B\sigma\upsilon\nu\Gamma.$$

4. $\boldsymbol{\eta\mu^2 2A+\eta\mu^2 2B+\eta\mu^2 2\Gamma=2-2\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma.}$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu^2 2A+\eta\mu^2 2B+\eta\mu^2 2\Gamma=\frac{1-\sigma\upsilon\nu 4A}{2}+\frac{1-\sigma\upsilon\nu 4B}{2}+\frac{1-\sigma\upsilon\nu 4\Gamma}{2}=$$
$$=\frac{3}{2}-\frac{1}{2}[\sigma\upsilon\nu 4A+\sigma\upsilon\nu 4B+\sigma\upsilon\nu 4\Gamma]=$$
$$=\frac{3}{2}-\frac{1}{2}[2\sigma\upsilon\nu(2A+2B)\sigma\upsilon\nu(2A-2B)+2\sigma\upsilon\nu^2 2\Gamma-1]$$
$$=\frac{3}{2}+\frac{1}{2}-\sigma\upsilon\nu(2A+2B)\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu^2 2\Gamma$$
$$=2-\sigma\upsilon\nu 2\Gamma\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu^2 2\Gamma=2-\sigma\upsilon\nu 2\Gamma[\sigma\upsilon\nu(2A-2B)+\sigma\upsilon\nu(2A+2B)]$$
$$=2-2\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma.$$

5. $\boldsymbol{\sigma\upsilon\nu^2 2A+\sigma\upsilon\nu^2 2B+\sigma\upsilon\nu^2 2\Gamma=1+2\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma.}$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu^2 2A+\sigma\upsilon\nu^2 2B+\sigma\upsilon\nu^2 2\Gamma=3-(\eta\mu^2 2A+\eta\mu^2 2B+\eta\mu^2 2\Gamma)$$
$$=3-(2-2\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma)=$$
$$=1+2\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma.$$

6. $\boldsymbol{\eta\mu(B+\Gamma-A)+\eta\mu(\Gamma+A-B)+\eta\mu(A+B-\Gamma)=4\eta\mu A\eta\mu B\eta\mu\Gamma.}$

Λύσις. Εἶναι : $\eta\mu(B+\Gamma-A)=\eta\mu(\pi-A-A)=\eta\mu(\pi-2A)=\eta\mu 2A.$
καὶ διὰ κυκλικῆς ἐναλλαγῆς τῶν A, B, Γ, θὰ ἔχωμεν :

$$\Sigma\,\eta\mu(B+\Gamma-A)=\eta\mu 2A+\eta\mu 2B+\eta\mu 2\Gamma=4\eta\mu A\eta\mu B\eta\mu\Gamma.$$

91. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $\boldsymbol{\eta\mu 4A+\eta\mu 4B+\eta\mu 4\Gamma=-4\eta\mu 2A\eta\mu 2B\eta\mu 2\Gamma.}$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu 4A+\eta\mu 4B+\eta\mu 4\Gamma=2\eta\mu(2A+2B)\sigma\upsilon\nu(2A-2B)+2\eta\mu 2\Gamma\sigma\upsilon\nu 2\Gamma$$
$$=-2\eta\mu 2\Gamma\sigma\upsilon\nu(2A-2B)+2\eta\mu 2\Gamma\sigma\upsilon\nu 2\Gamma$$
$$=-2\eta\mu 2\Gamma[\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu 2\Gamma]$$
$$=-2\eta\mu 2\Gamma[\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu(2A+2B)]$$
$$=-4\eta\mu 2A\eta\mu 2B\eta\mu 2\Gamma.$$

2. $\eta\mu 4A+\eta\mu 4B-\eta\mu 4\Gamma=-4\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\eta\mu 2\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu 4A+\eta\mu 4B-\eta\mu 4\Gamma=2\eta\mu(2A+2B)\sigma\upsilon\nu(2A-2B)-2\eta\mu 2\Gamma\sigma\upsilon\nu 2\Gamma=$$
$$=-2\eta\mu 2\Gamma\sigma\upsilon\nu(2A-2B)-2\eta\mu 2\Gamma\sigma\upsilon\nu 2\Gamma=$$
$$=-2\eta\mu 2\Gamma[\sigma\upsilon\nu 2A-2B)+\sigma\upsilon\nu 2\Gamma]=$$
$$=-2\eta\mu 2\Gamma[\sigma\upsilon\nu(2A-2B)+\sigma\upsilon\nu(2A+2B)]=$$
$$=-4\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\eta\mu 2\Gamma.$$

3. $\sigma\upsilon\nu 4A+\sigma\upsilon\nu 4B+\sigma\upsilon\nu 4\Gamma=-1+4\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu 4A+\sigma\upsilon\nu 4B+\sigma\upsilon\nu 4\Gamma=2\sigma\upsilon\nu(2A+2B)\sigma\upsilon\nu(2A-2B)+2\sigma\upsilon\nu^2 2\Gamma-1=$$
$$=-1+2\sigma\upsilon\nu 2\Gamma\sigma\upsilon\nu(2A-2B)+2\sigma\upsilon\nu^2 2\Gamma=$$
$$=-1+2\sigma\upsilon\nu 2\Gamma[\sigma\upsilon\nu(2A-2B)+\sigma\upsilon\nu(2A+2B)]$$
$$=-1+4\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma.$$

4. $\sigma\upsilon\nu 4A+\sigma\upsilon\nu 4B-\sigma\upsilon\nu 4\Gamma=1+4\eta\mu 2A\eta\mu 2B\sigma\upsilon\nu 2\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu 4A+\sigma\upsilon\nu 4B-\sigma\upsilon\nu 4\Gamma=2\sigma\upsilon\nu(2A+2B)\sigma\upsilon\nu(2A-2B)+1-2\sigma\upsilon\nu^2 2\Gamma=$$
$$=1+2\sigma\upsilon\nu 2\Gamma\sigma\upsilon\nu(2A-2B)-2\sigma\upsilon\nu^2 2\Gamma=$$
$$=1+2\sigma\upsilon\nu 2\Gamma[\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu 2\Gamma]=$$
$$=1+2\sigma\upsilon\nu 2\Gamma[\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu(2A+2B)]=$$
$$=1+4\eta\mu 2A\eta\mu 2B\sigma\upsilon\nu 2\Gamma.$$

92. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

1. $\eta\mu^2\frac{A}{2}+\eta\mu^2\frac{B}{2}+\eta\mu^2\frac{\Gamma}{2}=1-2\eta\mu\frac{\Lambda}{2}\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu^2\frac{A}{2}+\eta\mu^2\frac{B}{2}+\eta\mu^2\frac{\Gamma}{2}=\frac{1-\sigma\upsilon\nu A}{2}+\frac{1-\sigma\upsilon\nu B}{2}+\frac{1-\sigma\upsilon\nu\Gamma}{2}=$$
$$=\frac{3}{2}-\frac{1}{2}(\sigma\upsilon\nu A+\sigma\upsilon\nu B+\sigma\upsilon\nu\Gamma)=\frac{3}{2}-\frac{1}{2}\left(1+4\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}\right),$$
$$=\frac{3}{2}-\frac{1}{2}-2\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}=1-2\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}.$$

2. $$\eta\mu^2\frac{\mathbf{A}}{\mathbf{2}}+\eta\mu^2\frac{\mathbf{B}}{\mathbf{2}}-\eta\mu^2\frac{\mathbf{\Gamma}}{\mathbf{2}}=\mathbf{1}-\mathbf{2}\sigma\upsilon\nu\frac{\mathbf{A}}{\mathbf{2}}\,\sigma\upsilon\nu\frac{\mathbf{B}}{\mathbf{2}}\,\eta\mu\frac{\mathbf{\Gamma}}{\mathbf{2}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu^2\frac{A}{2}+\eta\mu^2\frac{B}{2}-\eta\mu^2\frac{\Gamma}{2}=\frac{1-\sigma\upsilon\nu A}{2}+\frac{1-\sigma\upsilon\nu B}{2}-\frac{1-\sigma\upsilon\nu\Gamma}{2}=$$

$$=\frac{1}{2}-\frac{1}{2}\,[\sigma\upsilon\nu A+\sigma\upsilon\nu B-\sigma\upsilon\nu\Gamma]=$$

$$=\frac{1}{2}-\frac{1}{2}\left[-1+4\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\eta\mu\frac{\Gamma}{2}\right]=$$

$$=1-2\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\eta\mu\frac{\Gamma}{2}.$$

3. $$\frac{\eta\mu\mathbf{A}+\eta\mu\mathbf{B}-\eta\mu\mathbf{\Gamma}}{\eta\mu\mathbf{A}-\eta\mu\mathbf{B}+\eta\mu\mathbf{\Gamma}}=\varepsilon\varphi\frac{\mathbf{B}}{\mathbf{2}}\,\sigma\varphi\frac{\mathbf{\Gamma}}{\mathbf{2}}.$$

Λύσις. Εἶναι:

$$\frac{\eta\mu A+\eta\mu B-\eta\mu\Gamma}{\eta\mu A-\eta\mu B+\eta\mu\Gamma}=\frac{4\eta\mu\frac{A}{2}\,\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}}{4\eta\mu\frac{A}{2}\,\eta\mu\frac{\Gamma}{2}\,\sigma\upsilon\nu\frac{B}{2}}=\varepsilon\varphi\frac{B}{2}\,\sigma\varphi\frac{\Gamma}{2}.$$

4. $$\frac{\sigma\upsilon\nu\mathbf{A}+\sigma\upsilon\nu\mathbf{B}+\sigma\upsilon\nu\mathbf{\Gamma}-\mathbf{1}}{\sigma\upsilon\nu\mathbf{A}+\sigma\upsilon\nu\mathbf{B}-\sigma\upsilon\nu\mathbf{\Gamma}+\mathbf{1}}=\varepsilon\varphi\frac{\mathbf{A}}{\mathbf{2}}\,\varepsilon\varphi\frac{\mathbf{B}}{\mathbf{2}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\sigma\upsilon\nu A+\sigma\upsilon\nu B+\sigma\upsilon\nu\Gamma-1}{\sigma\upsilon\nu A+\sigma\upsilon\nu B-\sigma\upsilon\nu\Gamma+1}=\frac{4\eta\mu\frac{A}{2}\,\eta\mu\frac{B}{2}\,\eta\mu\frac{\Gamma}{2}}{4\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\eta\mu\frac{\Gamma}{2}}=\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2}.$$

5. $$\frac{\eta\mu\mathbf{2A}+\eta\mu\mathbf{2B}+\eta\mu\mathbf{2\Gamma}}{\eta\mu\mathbf{2A}+\eta\mu\mathbf{2B}-\eta\mu\mathbf{2\Gamma}}=\varepsilon\varphi\mathbf{A}\,\varepsilon\varphi\mathbf{B}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\eta\mu 2A+\eta\mu 2B+\eta\mu 2\Gamma}{\eta\mu 2A+\eta\mu 2B-\eta\mu 2\Gamma}=\frac{4\eta\mu A\,\eta\mu B\,\eta\mu\Gamma}{4\sigma\upsilon\nu A\,\sigma\upsilon\nu B\,\eta\mu\Gamma}=\varepsilon\varphi A\,\varepsilon\varphi B.$$

6. $$\frac{\eta\mu\mathbf{2A}+\eta\mu\mathbf{2B}+\eta\mu\mathbf{2\Gamma}}{\eta\mu\mathbf{A}+\eta\mu\mathbf{B}+\eta\mu\mathbf{\Gamma}}=8\eta\mu\frac{\mathbf{A}}{\mathbf{2}}\,\eta\mu\frac{\mathbf{B}}{\mathbf{2}}\,\eta\mu\frac{\mathbf{\Gamma}}{\mathbf{2}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\eta\mu 2A+\eta\mu 2B+\eta\mu 2\Gamma}{\eta\mu A+\eta\mu B+\eta\mu\Gamma}=\frac{4\eta\mu A\,\eta\mu B\,\eta\mu\Gamma}{4\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}}=$$

$$=\frac{2\eta\mu\frac{A}{2}\sigma\upsilon\nu\frac{A}{2}\cdot 2\eta\mu\frac{B}{2}\sigma\upsilon\nu\frac{B}{2}\cdot 2\eta\mu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{\Gamma}{2}}{\sigma\upsilon\nu\frac{A}{2}\sigma\upsilon\nu\frac{B}{2}\sigma\upsilon\nu\frac{\Gamma}{2}}=$$

$$=8\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}.$$

7. $$\frac{\sigma\varphi B+\sigma\varphi\Gamma}{\varepsilon\varphi B+\varepsilon\varphi\Gamma}+\frac{\sigma\varphi\Gamma+\sigma\varphi A}{\varepsilon\varphi\Gamma+\varepsilon\varphi A}+\frac{\sigma\varphi A+\sigma\varphi B}{\varepsilon\varphi A+\varepsilon\varphi B}=1.$$

Λύσις. Τὸ πρῶτον κλάσμα γράφεται διαδοχικῶς:

$$\frac{\sigma\varphi B+\sigma\varphi\Gamma}{\varepsilon\varphi B+\varepsilon\varphi\Gamma}=\frac{\sigma\varphi B+\sigma\varphi\Gamma}{\frac{1}{\sigma\varphi B}+\frac{1}{\sigma\varphi\Gamma}}=\frac{(\sigma\varphi B+\sigma\varphi\Gamma)\sigma\varphi B\sigma\varphi\Gamma}{\sigma\varphi B+\sigma\varphi\Gamma}=\sigma\varphi B\sigma\varphi\Gamma.$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς, τὸ α΄ μέλος γράφεται:

$$\sigma\varphi B\sigma\varphi\Gamma+\sigma\varphi\Gamma\sigma\varphi A+\sigma\varphi A\sigma\varphi B=1. \qquad \text{(ἄσκησις 15,2)}$$

8. $$\frac{\varepsilon\varphi A+\varepsilon\varphi B+\varepsilon\varphi\Gamma}{(\eta\mu A+\eta\mu B+\eta\mu\Gamma)^2}=\frac{\varepsilon\varphi\frac{A}{2}\varepsilon\varphi\frac{B}{2}\varepsilon\varphi\frac{\Gamma}{2}}{2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\varepsilon\varphi A+\varepsilon\varphi B+\varepsilon\varphi\Gamma}{(\eta\mu A+\eta\mu B+\eta\mu\Gamma)^2}=\frac{\varepsilon\varphi A\varepsilon\varphi B\varepsilon\varphi\Gamma}{16\sigma\upsilon\nu^2\frac{A}{2}\sigma\upsilon\nu^2\frac{B}{2}\sigma\upsilon\nu^2\frac{\Gamma}{2}}=$$

$$\frac{\eta\mu A\eta\mu B\eta\mu\Gamma}{\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\cdot 16\sigma\upsilon\nu^2\frac{A}{2}\sigma\upsilon\nu^2\frac{B}{2}\sigma\upsilon\nu^2\frac{\Gamma}{2}}=$$

$$=\frac{2\eta\mu\frac{A}{2}\sigma\upsilon\nu\frac{A}{2}\cdot 2\eta\mu\frac{B}{2}\sigma\upsilon\nu\frac{B}{2}\cdot 2\eta\mu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{\Gamma}{2}}{4\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\sigma\upsilon\nu^2\frac{A}{2}\sigma\upsilon\nu^2\frac{B}{2}\sigma\upsilon\nu^2\frac{\Gamma}{2}}=$$

$$=\frac{\eta\mu\frac{A}{2}\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}}{2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\sigma\upsilon\nu\frac{A}{2}\sigma\upsilon\nu\frac{B}{2}\sigma\upsilon\nu\frac{\Gamma}{2}}=\frac{\varepsilon\varphi\frac{A}{2}\varepsilon\varphi\frac{B}{2}\varepsilon\varphi\frac{\Gamma}{2}}{2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma}.$$

93. Ἐὰν $A+B+\Gamma=180^0$, *νὰ γίνουν γινόμενα αἱ παραστάσεις:*

1, $$\eta\mu 3A+\eta\mu 3B+\eta\mu 3\Gamma.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$K=\eta\mu 3A+\eta\mu 3B+\eta\mu 3\Gamma=2\eta\mu\frac{3A+3B}{2}\sigma\upsilon\nu\frac{3A-3B}{2}+2\eta\mu\frac{3\Gamma}{2}\sigma\upsilon\nu\frac{3\Gamma}{2} \qquad (1)$$

Ἐπειδὴ $A+B+\Gamma=180^0 \Longrightarrow 3A+3B+3\Gamma=540^0$ ἢ

$$\frac{3A+3B}{2}+\frac{3\Gamma}{2}=270^0 \Longrightarrow \eta\mu\frac{3A+3B}{2}=\eta\mu\left(270^0-\frac{3\Gamma}{2}\right)=-\sigma\upsilon\nu\frac{3\Gamma}{2}$$

ὁπότε θὰ ἔχωμεν:

$$K=-2\sigma\upsilon\nu\frac{3\Gamma}{2}\sigma\upsilon\nu\frac{3A-3B}{2}+2\eta\mu\frac{3\Gamma}{2}\sigma\upsilon\nu\frac{3\Gamma}{2}=$$

$$=-2\sigma\upsilon\nu\frac{3\Gamma}{2}\left[\sigma\upsilon\nu\frac{3A-3B}{2}-\eta\mu\frac{3\Gamma}{2}\right]=-2\sigma\upsilon\nu\frac{3\Gamma}{2}\left[\sigma\upsilon\nu\frac{3A-3B}{2}+\sigma\upsilon\nu\frac{3A+3B}{2}\right]$$

$$=-4\sigma\upsilon\nu\frac{3A}{2}\sigma\upsilon\nu\frac{3B}{2}\sigma\upsilon\nu\frac{3\Gamma}{2}.$$

2. $\quad \boldsymbol{\eta\mu 6A+\eta\mu 6B+\eta\mu 6\Gamma.}$

Λύσις. Ἐπειδὴ $A+B+\Gamma=180^0 \Longrightarrow 3A+3B=540^0-3\Gamma$

καὶ ἄρα $\eta\mu(3A+3B)=\eta\mu(540^0-3\Gamma)=\eta\mu[360^0+180^0-3\Gamma]=$

$$=\eta\mu(180^0-3\Gamma)=\eta\mu 3\Gamma$$

καὶ $\sigma\upsilon\nu\, 3\Gamma=\sigma\upsilon\nu[540^0-(3A+3B)]=\sigma\upsilon\nu[360^0+180^0-(3A+3B)]=$

$$=\sigma\upsilon\nu\,[180^0-(3A+3B)]=-\sigma\upsilon\nu(3A+3B).$$

Ἄρα θὰ ἔχωμεν διαδοχικῶς:

$$\eta\mu 6A+\eta\mu 6B+\eta\mu 6\Gamma=2\eta\mu(3A+3B)\sigma\upsilon\nu(3A-3B)+2\eta\mu 3\Gamma\sigma\upsilon\nu 3\Gamma$$

$$=2\eta\mu 3\Gamma\sigma\upsilon\nu(3A-3B)+2\eta\mu 3\Gamma\sigma\upsilon\nu 3\Gamma$$

$$=2\eta\mu 3\Gamma[\sigma\upsilon\nu(3A-3B)-\sigma\upsilon\nu(3A+3B)]$$

$$=4\eta\mu 3A\eta\mu 3B\eta\mu 3\Gamma.$$

3. $\quad \boldsymbol{\varepsilon\varphi(kA)+\varepsilon\varphi(kB)+\varepsilon\varphi(k\Gamma)}, \quad k\in \mathbf{N}$

Λύσις. Ἐπειδὴ $A+B+\Gamma=180^0 \Longrightarrow kA+kB=k\cdot 180^0-k\Gamma$

ἕπεται ὅτι: $\varepsilon\varphi(kA+kB)=\varepsilon\varphi(k180^0-k\Gamma)=-\varepsilon\varphi(k\Gamma)$

ἢ $$\frac{\varepsilon\varphi(kA)+\varepsilon\varphi(kB)}{1-\varepsilon\varphi(kA)\varepsilon\varphi(kB)}=-\varepsilon\varphi(k\Gamma)$$ ἐξ οὗ:

$$\varepsilon\varphi(kA)+\varepsilon\varphi(kB)+\varepsilon\varphi(k\Gamma)=\varepsilon\varphi(kA)\varepsilon\varphi(kB)\varepsilon\varphi(k\Gamma). \qquad (1)$$

4. $\quad \boldsymbol{\sigma\varphi(kA)\sigma\varphi(kB)+\sigma\varphi(kB)\sigma\varphi(k\Gamma)+\sigma\varphi(k\Gamma)\sigma\varphi(kA)=1.}$

Λύσις. Ἐκ τῆς $kA+kB=k180^0-k\Gamma$, ἕπεται ὅτι:

$\sigma\varphi(kA+kB)=\sigma\varphi(k\cdot 180^0-k\Gamma)=-\sigma\varphi(k\Gamma)$, ἢ

$$\frac{\sigma\varphi(kA)\sigma\varphi(kB)-1}{\sigma\varphi(kA)+\sigma\varphi(kB)}=-\sigma\varphi(k\Gamma),$$ ἐξ οὗ:

$$\sigma\varphi(kA)\sigma\varphi(kB)+\sigma\varphi(kB)\sigma\varphi(k\Gamma)+\sigma\varphi(k\Gamma)\sigma\varphi(kA)=1.$$

94. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

1. $$\boldsymbol{\eta\mu\frac{A}{2}+\eta\mu\frac{B}{2}+\eta\mu\frac{\Gamma}{2}=1+4\eta\mu\frac{\pi-A}{4}\eta\mu\frac{\pi-B}{4}\eta\mu\frac{\pi-\Gamma}{4}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu\frac{A}{2}+\eta\mu\frac{B}{2}+\eta\mu\frac{\Gamma}{2}=2\eta\mu\frac{A+B}{4}\sigma\upsilon\nu\frac{A-B}{4}+\sigma\upsilon\nu\frac{A+B}{2}=$$

$$=2\eta\mu\frac{\pi-\Gamma}{4}\sigma\upsilon\nu\frac{A-B}{4}+1-2\eta\mu^2\frac{\pi-\Gamma}{4}$$

$$=1+2\eta\mu\frac{\pi-\Gamma}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}-\eta\mu\frac{\pi-\Gamma}{4}\right]=1+2\eta\mu\frac{\pi-\Gamma}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}-\eta\mu\frac{A+B}{4}\right]$$

$$=1+2\eta\mu\frac{\pi-\Gamma}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}-\sigma\upsilon\nu\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$

$$=1+2\eta\mu\frac{\pi-\Gamma}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}-\sigma\upsilon\nu\frac{2\pi-A-B}{4}\right]=$$

$$=1+2\eta\mu\frac{\pi-\Gamma}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}-\sigma\upsilon\nu\frac{A+B+2\Gamma}{4}\right]=$$

$$=1+2\eta\mu\frac{\pi-\Gamma}{4}\cdot 2\eta\mu\frac{\pi-B}{4}\eta\mu\frac{\pi-A}{4}=1+4\eta\mu\frac{\pi-A}{4}\eta\mu\frac{\pi-B}{4}\eta\mu\frac{\pi-\Gamma}{4}$$

2. $$\sigma\upsilon\nu\frac{A}{2}+\sigma\upsilon\nu\frac{B}{2}+\sigma\upsilon\nu\frac{\Gamma}{2}=4\sigma\upsilon\nu\frac{B+\Gamma}{4}\sigma\upsilon\nu\frac{\Gamma+A}{4}\sigma\upsilon\nu\frac{A+B}{4}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu\frac{A}{2}+\sigma\upsilon\nu\frac{B}{2}+\sigma\upsilon\nu\frac{\Gamma}{2}=2\sigma\upsilon\nu\frac{A+B}{4}\sigma\upsilon\nu\frac{A-B}{4}+\eta\mu\frac{A+B}{2}=$$

$$=2\sigma\upsilon\nu\frac{A+B}{4}\sigma\upsilon\nu\frac{A-B}{4}+2\eta\mu\frac{A+B}{4}\sigma\upsilon\nu\frac{A+B}{4}=$$

$$=2\sigma\upsilon\nu\frac{A+B}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}+\eta\mu\frac{A+B}{4}\right]$$

$$=2\sigma\upsilon\nu\frac{A+B}{4}\left[\sigma\upsilon\nu\frac{A-B}{4}+\sigma\upsilon\nu\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$

$$=2\sigma\upsilon\nu\frac{A+B}{4}\cdot 2\sigma\upsilon\nu\frac{A-B+2\pi-A-B}{8}\sigma\upsilon\nu\frac{A-B-2\pi+A+B}{8}$$

$$=4\sigma\upsilon\nu\frac{A+B}{4}\sigma\upsilon\nu\frac{\pi-B}{4}\sigma\upsilon\nu\frac{\pi-A}{4}$$

$$=4\sigma\upsilon\nu\frac{A+B}{4}\sigma\upsilon\nu\frac{\Gamma+A}{4}\sigma\upsilon\nu\frac{B+\Gamma}{4}=4\sigma\upsilon\nu\frac{B+\Gamma}{4}\sigma\upsilon\nu\frac{\Gamma+A}{4}\sigma\upsilon\nu\frac{A+B}{4}.$$

3. $$\sigma\upsilon\nu\frac{A}{2}-\sigma\upsilon\nu\frac{B}{2}+\sigma\upsilon\nu\frac{\Gamma}{2}=4\sigma\upsilon\nu\frac{\pi+A}{4}\sigma\upsilon\nu\frac{\pi-B}{4}\sigma\upsilon\nu\frac{\pi+\Gamma}{4}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\sigma\upsilon\nu\,\frac{A}{2}-\sigma\upsilon\nu\,\frac{B}{2}+\sigma\upsilon\nu\,\frac{\Gamma}{2}=2\eta\mu\,\frac{A+B}{4}\,\eta\mu\,\frac{B-A}{4}+\eta\mu\,\frac{A+B}{2}=$$

$$=2\eta\mu\,\frac{A+B}{4}\,\eta\mu\frac{B-A}{4}+2\eta\mu\,\frac{A+B}{4}\,\sigma\upsilon\nu\,\frac{A+B}{4}$$

$$=2\eta\mu\,\frac{A+B}{4}\left[\eta\mu\,\frac{B-A}{4}+\sigma\upsilon\nu\,\frac{A+B}{4}\right]=$$

$$=2\eta\mu\frac{A+B}{4}\left[\eta\mu\,\frac{B-A}{4}+\eta\mu\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]=$$

$$=2\eta\mu\,\frac{A+B}{4}\cdot 2\eta\mu\,\frac{B-A+2\pi-A-B}{8}\,\sigma\upsilon\nu\,\frac{B-A-2\pi+A+B}{8}$$

$$=4\eta\mu\,\frac{A+B}{4}\,\eta\mu\,\frac{\pi-A}{4}\,\sigma\upsilon\nu\frac{\pi-B}{4}=4\sigma\upsilon\nu\frac{\pi+\Gamma}{4}\,\sigma\upsilon\nu\frac{\pi+A}{4}\,\sigma\upsilon\nu\frac{\pi-B}{4}$$

διότι $$\frac{A+B}{4}+\frac{\pi+\Gamma}{4}=\frac{\pi+A+B+\Gamma}{4}=\frac{\pi+\pi}{4}=\frac{\pi}{2}\Rightarrow$$

$$\Rightarrow\eta\mu\,\frac{A+B}{4}=\sigma\upsilon\nu\,\frac{\pi+\Gamma}{4}$$

καὶ $$\frac{\pi-A}{4}+\frac{\pi+A}{4}=\frac{2\pi}{4}=\frac{\pi}{2}\Rightarrow\eta\mu\,\frac{\pi-A}{4}=\sigma\upsilon\nu\,\frac{\pi+A}{4}.$$

4. $$\eta\mu\,\frac{A}{2}+\eta\mu\,\frac{B}{2}-\eta\mu\,\frac{\Gamma}{2}=-1+4\,\sigma\upsilon\nu\frac{\pi-A}{4}\,\sigma\upsilon\nu\frac{\pi-B}{4}\,\eta\mu\frac{\pi-\Gamma}{4}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu\,\frac{A}{2}+\eta\mu\,\frac{B}{2}-\eta\mu\,\frac{\Gamma}{2}=2\eta\mu\,\frac{A+B}{4}\,\sigma\upsilon\nu\,\frac{A-B}{4}-\sigma\upsilon\nu\,\frac{A+B}{2}=$$

$$=2\eta\mu\,\frac{A+B}{4}\,\sigma\upsilon\nu\,\frac{A-B}{4}-1+2\eta\mu^2\,\frac{A+B}{4}=$$

$$=-1+2\eta\mu\,\frac{A+B}{4}\left[\sigma\upsilon\nu\,\frac{A-B}{4}+\eta\mu\,\frac{A+B}{4}\right]=$$

$$=-1+2\eta\mu\,\frac{A+B}{4}\left[\sigma\upsilon\nu\,\frac{A-B}{4}+\sigma\upsilon\nu\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]=$$

$$=-1+2\eta\mu\,\frac{A+B}{4}\cdot 2\sigma\upsilon\nu\,\frac{A-B+2\pi-A-B}{8}\,\sigma\upsilon\nu\,\frac{A-B-2\pi+A+B}{8}=$$

$$=-1+4\eta\mu\,\frac{A+B}{4}\,\sigma\upsilon\nu\,\frac{\pi-B}{4}\,\sigma\upsilon\nu\,\frac{\pi-A}{4}=-1+4\sigma\upsilon\nu\,\frac{\pi-A}{4}\,\sigma\upsilon\nu\,\frac{\pi-\Gamma}{4}\,\eta\mu\,\frac{\pi-\Gamma}{4}.$$

5. $$\eta\mu^2\,\frac{A}{4}+\eta\mu^2\frac{B}{4}+\eta\mu^2\frac{\Gamma}{4}=\frac{3}{2}-2\sigma\upsilon\nu\frac{\pi-A}{4}\,\sigma\upsilon\nu\frac{\pi-B}{4}\,\sigma\upsilon\nu\frac{\pi-\Gamma}{4}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu^2\,\frac{A}{4}+\eta\mu^2\,\frac{B}{4}+\eta\mu^2\,\frac{\Gamma}{4}=\frac{1-\sigma\upsilon\nu\frac{A}{2}}{2}+\frac{1-\sigma\upsilon\nu\frac{B}{2}}{2}+\frac{1-\sigma\upsilon\nu\frac{\Gamma}{2}}{2}=$$

$$=\frac{3}{2}-\frac{1}{2}\left[\text{συν}\frac{A}{2}+\text{συν}\frac{B}{2}+\text{συν}\frac{\Gamma}{2}\right]=$$

$$=\frac{3}{2}-\frac{1}{2}\cdot 4\text{συν}\frac{B+\Gamma}{4}\text{συν}\frac{\Gamma+A}{4}\text{συν}\frac{A+B}{4}=$$

$$=\frac{3}{2}-2\text{συν}\frac{\pi-A}{4}\text{συν}\frac{\pi-B}{4}\text{συν}\frac{\pi-\Gamma}{4}.$$

6. $\mathbf{ημ^2\frac{A}{4}+ημ^2\frac{B}{4}-ημ^2\frac{\Gamma}{4}=\frac{1}{2}-2ημ\frac{\pi-A}{4}ημ\frac{\pi-B}{4}συν\frac{\pi-\Gamma}{4}}$.

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{ημ}^2\frac{A}{4}+\text{ημ}^2\frac{B}{4}-\text{ημ}^2\frac{\Gamma}{4}=\frac{1-\text{συν}\frac{A}{2}}{2}+\frac{1-\text{συν}\frac{B}{2}}{2}-\frac{1-\text{συν}\frac{\Gamma}{2}}{2}=$$

$$=\frac{1}{2}-\frac{1}{2}\left[\text{συν}\frac{A}{2}+\text{συν}\frac{B}{2}-\text{συν}\frac{\Gamma}{2}\right]=$$

$$=\frac{1}{2}-\frac{1}{2}\cdot 4\text{συν}\frac{\pi+A}{4}\text{συν}\frac{\pi+B}{4}\text{συν}\frac{\pi-\Gamma}{4}=$$

$$=\frac{1}{2}-2\text{ημ}\frac{\pi-A}{4}\text{ημ}\frac{\pi-B}{4}\text{συν}\frac{\pi-\Gamma}{4}.$$

7. $\mathbf{συν^2\frac{A}{4}+συν^2\frac{B}{4}+συν^2\frac{\Gamma}{4}=\frac{3}{2}+2συν\frac{\pi-A}{4}συν\frac{\pi-B}{4}συν\frac{\pi-\Gamma}{4}}$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{συν}^2\frac{A}{4}+\text{συν}^2\frac{B}{4}+\text{συν}^2\frac{\Gamma}{4}=\frac{1+\text{συν}\frac{A}{2}}{2}+\frac{1+\text{συν}\frac{B}{2}}{2}+\frac{1+\text{συν}\frac{\Gamma}{2}}{2}=$$

$$=\frac{3}{2}+\frac{1}{2}\left[\text{συν}\frac{A}{2}+\text{συν}\frac{B}{2}+\text{συν}\frac{\Gamma}{2}\right]=$$

$$=\frac{3}{2}+\frac{1}{2}\cdot 4\text{συν}\frac{B+\Gamma}{4}\text{συν}\frac{\Gamma+A}{4}\text{συν}\frac{A+B}{4}=$$

$$=\frac{3}{2}+2\text{συν}\frac{\pi-A}{4}\text{συν}\frac{\pi-B}{4}\text{συν}\frac{\pi-\Gamma}{4}.$$

8, $\mathbf{συν^2\frac{A}{4}+συν^2\frac{B}{4}-συν^2\frac{\Gamma}{4}=\frac{1}{2}+2ημ\frac{\pi-A}{4}ημ\frac{\pi-B}{4}συν\frac{\pi-\Gamma}{4}}$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\text{συν}^2\frac{A}{4}+\text{συν}^2\frac{B}{4}-\text{συν}^2\frac{\Gamma}{4}=\frac{1+\text{συν}\frac{A}{2}}{2}+\frac{1+\text{συν}\frac{B}{2}}{2}-\frac{1+\text{συν}\frac{\Gamma}{2}}{2}=$$

$$= \frac{1}{2} + \frac{1}{2}\left[\text{συν}\,\frac{A}{2} + \text{συν}\,\frac{B}{2} - \text{συν}\,\frac{\Gamma}{2} \right] =$$

$$= \frac{1}{2} + \frac{1}{2} \cdot 4\text{συν}\,\frac{\pi + A}{4}\,\text{συν}\,\frac{\pi + B}{4}\,\text{συν}\,\frac{\pi - \Gamma}{4} =$$

$$= \frac{1}{2} + 2\text{ημ}\,\frac{\pi - A}{4}\,\text{ημ}\,\frac{\pi - B}{4}\,\text{συν}\,\frac{\pi - \Gamma}{4}.$$

95. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

1. **Σ ημΑσυνΒσυνΓ=ημΑημΒημΓ.**

Λύσις. Ἔχομεν διαδοχικῶς:

Σ ημΑσυνΒσυνΓ = ημΑσυνΒσυνΓ + ημΒσυνΓσυνΑ + ημΓσυνΑσυνΒ
= συνΓ(ημΑσυνΒ + ημΒσυνΑ) + ημΓσυνΑσυνΒ
= συνΓ · ημ(Α + Β)ημΓσυνΑσυνΒ
= συνΓημΓ + ημΓσυνΑσυνΒ
= ημΓ[συνΓ + συνΑσυνΒ] = ημΓ[− συν(Α + Β) + συνΑσυνΒ] =
= ημΓ(− συνΑσυνΒ + ημΑημΒ + συνΑσυνΒ)
= ημΑημΒημΓ.

2. **Σ συνΑημΒημΓ=1 + συνΑσυνΒσυνΓ.**

Λύσις. Ἔχομεν διαδοχικῶς:

Σ συνΑημΒημΓ = συνΑημΒημΓ + συνΒημΓημΑ + συνΓημΑημΒ =
= ημΓ(συνΑημΒ + συνΒημΑ) + συνΓημΑημΒ
= ημΓ · ημ(Α + Β) + συνΓημΑημΒ
= ημΓ · ημΓ + συνΓημΑημΒ = $\text{ημ}^2\Gamma$ + συνΓημΑημΒ
= 1 − $\text{συν}^2\Gamma$ + συνΓημΑημΒ = 1 − συνΓ[συνΓ − ημΑημΒ]
= 1 − συνΓ[− συν(Α + Β) − ημΑημΒ]
= 1 − συνΓ[− συνΑσυνΒ + ημΑημΒ − ημΑημΒ]
= 1 + συνΑσυνΒσυνΓ.

3. **Σ ημΑσυν(Β−Γ)=4ημΑημΒημΓ.**

Λύσις. Ἔχομεν διαδοχικῶς:

Σ ημΑσυν(Β − Γ) = ημΑσυν(Β − Γ) + ημΒσυν(Γ − Α) + ημΓσυν(Α − Β) =
= ημΑ(συνΒσυνΓ + ημΒημΓ) + ημΒ(συνΓσυνΑ + ημΓημΑ) +
+ ημΓ(συνΑσυνΒ + ημΑημΒ) =
= (ημΑσυνΒσυνΓ + ημΒσυνΓσυνΑ + ημΓσυνΑσυνΒ) + 3ημΑημΒημΓ =
= ημΑημΒημΓ + 3ημΑημΒημΓ = 4ημΑημΒημΓ.

4. $\Sigma\ \sigma\upsilon\nu A\sigma\upsilon\nu(B-\Gamma)=1+4\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma.$

Λύσις. Ἔχομεν διαδοχικῶς:

$=\Sigma\ \sigma\upsilon\nu A\sigma\upsilon\nu(B-\Gamma)=\sigma\upsilon\nu A\sigma\upsilon\nu(B-\Gamma)+\sigma\upsilon\nu B\sigma\upsilon\nu(\Gamma-A)+\sigma\upsilon\nu\Gamma\sigma\upsilon\nu(A-B)=$

$=-\sigma\upsilon\nu(B+\Gamma)\sigma\upsilon\nu(B-\Gamma)-\sigma\upsilon\nu(\Gamma+A)\sigma\upsilon\nu)\Gamma-A)-\sigma\upsilon\nu(A+B)\sigma\upsilon\nu(A-B)-$

$=\eta\mu^2\Gamma=\sigma\upsilon\nu^2B+\eta\mu^2A-\sigma\upsilon\nu^2\Gamma+\eta\mu^2B-\sigma\upsilon\nu^2A=$

$=-(\sigma\upsilon\nu^2A-\eta\mu^2A)-(\sigma\upsilon\nu^2B-\eta\mu^2B)-(\sigma\upsilon\nu^2\Gamma-\eta\mu^2\Gamma)=$

$=-\sigma\upsilon\nu 2A-\sigma\upsilon\nu 2B-\sigma\upsilon\nu 2\Gamma=-(\sigma\upsilon\nu 2A+\sigma\upsilon\nu 2B+\sigma\upsilon\nu 2\Gamma)=$

$=-(1-4\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma)=1+4\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma.$

96. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι.

1. $\Sigma\ \eta\mu 3A\eta\mu(B-\Gamma)=0.$

Λύσις. Ἔχομεν διαδοχικῶς:

$\Sigma\,\eta\mu 3A\eta\mu(B-\Gamma)=\eta\mu 3A\eta\mu(B-\Gamma)+\eta\mu 3B\eta\mu(\Gamma-A)+\eta\mu 3\Gamma\eta\mu(A-B)$

$=(3\eta\mu A-4\eta\mu^3A)\eta\mu(B-\Gamma)+\dots.$

$=3\eta\mu A\eta\mu(B-\Gamma)-4\eta\mu^3A\eta\mu(B-\Gamma)+\dots.$

$=3\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)-4\eta\mu^2A\cdot\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)+\dots.$

$=3(\eta\mu^2B-\eta\mu^2\Gamma)-4\eta\mu^2A(\eta\mu^2B-\eta\mu^2\Gamma)+\dots$

$=(\eta\mu^2B-\eta\mu^2\Gamma)(3-4\eta\mu^2A)+(\eta\mu^2\Gamma-\eta\mu^2A)(3-4\eta\mu^2B)+$

$+(\eta\mu^2A-\eta\mu^2B)(3-4\eta\mu^2\Gamma)=0.$

2. $\Sigma\ \eta\mu^3A\sigma\upsilon\nu(B-\Gamma)-3\eta\mu A\eta\mu B\eta\mu\Gamma=0.$

Λύσις. Ἔχομεν διαδοχικῶς:

$\eta\mu^3A\sigma\upsilon\nu(B-\Gamma)=\eta\mu^2A\eta\mu A\sigma\upsilon\nu(B-\Gamma)=\eta\mu^2A\eta\mu(B+\Gamma)\sigma\upsilon\nu(B-\Gamma)=$

$=\eta\mu^2A\cdot\frac{1}{2}(\eta\mu 2B+\eta\mu 2\Gamma)=\eta\mu^2A(\eta\mu B\sigma\upsilon\nu B+\eta\mu\Gamma\sigma\upsilon\nu\Gamma)=$

$=\eta\mu^2A\eta\mu B\sigma\upsilon\nu B+\eta\mu^2A\eta\mu\Gamma\sigma\upsilon\nu\Gamma.$

Ἄρα, διὰ κυκλικῆς ἐναλλαγῆς, θὰ εἶναι:

$\Sigma\,\eta\mu^3A\sigma\upsilon\nu(B-\Gamma)=\eta\mu^2A\eta\mu B\sigma\upsilon\nu B+\eta\mu^2A\eta\mu\Gamma\sigma\upsilon\nu\Gamma+$

$+\eta\mu^2B\eta\mu\Gamma\sigma\upsilon\nu\Gamma+\eta\mu^2B\eta\mu A\sigma\upsilon\nu A+\eta\mu^2\Gamma\eta\mu A\sigma\upsilon\nu A+\eta\mu^2\Gamma\eta\mu B\sigma\upsilon\nu B=$

$=\eta\mu A\eta\mu\Gamma(\eta\mu A\sigma\upsilon\nu\Gamma+\eta\mu\Gamma\sigma\upsilon\nu A)+\eta\mu B\eta\mu A(\eta\mu B\sigma\upsilon\nu A+\sigma\upsilon\nu B\eta\mu A)+$

$+\eta\mu\Gamma\eta\mu B(\eta\mu\Gamma\sigma\upsilon\nu B+\eta\mu B\sigma\upsilon\nu\Gamma)=$

$=\eta\mu A\eta\mu\Gamma\eta\mu(A+\Gamma)+\eta\mu B\eta\mu A\eta\mu(B+A)+\eta\mu\Gamma\eta\mu B\eta\mu(\Gamma+B)=$

$=\eta\mu A\eta\mu\Gamma\eta\mu B+\eta\mu B\eta\mu A\eta\mu\Gamma+\eta\mu\Gamma\eta\mu B\eta\mu A=3\eta\mu A\eta\mu B\eta\mu\Gamma.$ Ἄρα:

$\Sigma\,\eta\mu^3A\sigma\upsilon\nu(B-\Gamma)-3\eta\mu A\eta\mu B\eta\mu\Gamma=3\eta\mu A\eta\mu B\eta\mu\Gamma-3\eta\mu A\eta\mu B\eta\mu\Gamma=0.$

3. $\Sigma\ \sigma\upsilon\nu^3A\eta\mu(B-\Gamma)+\Pi\eta\mu(A-B)=0.$

Λύσις Γνωρίζομεν ὅτι: $\sigma\upsilon\nu 3A=4\sigma\upsilon\nu^3A-3\sigma\upsilon\nu A.$

Ἄρα: $\sigma\upsilon\nu^3A=\frac{1}{4}(\sigma\upsilon\nu 3A+3\sigma\upsilon\nu A),$ ὁπότε:

$$\sigma\upsilon\nu^3 A \eta\mu(B-\Gamma) = \frac{1}{4}(\sigma\upsilon\nu 3A + 3\sigma\upsilon\nu A)\eta\mu(B-\Gamma)$$

$$= \frac{1}{4}\sigma\upsilon\nu 3A\eta\mu(B-\Gamma) + \frac{3}{4}\sigma\upsilon\nu A\eta\mu(B-\Gamma)$$

$$= \frac{1}{8}\cdot 2\sigma\upsilon\nu 3A\eta\mu(B-\Gamma) + \frac{3}{8}\cdot 2\sigma\upsilon\nu A\eta\mu(B-\Gamma)$$

$$= \frac{1}{8}[\eta\mu(3A+B-\Gamma) - \eta\mu(3A-B+\Gamma)] + \frac{3}{8}[\eta\mu(A+B-\Gamma) - \eta\mu(A-B+\Gamma)].$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν A, B, Γ, λαμβάνομεν :

$$\Sigma\, \sigma\upsilon\nu^3 A\eta\mu(B-\Gamma) = \frac{1}{8}[\eta\mu(3A+B-\Gamma) - \eta\mu(3A-B+\Gamma) +$$

$$+\eta\mu(3B+\Gamma-A) - \eta\mu(3B-\Gamma+A) + \eta\mu(3\Gamma+A-B) - \eta\mu(3\Gamma-A+B)] +$$

$$+ \frac{3}{8}[\eta\mu(A+B-\Gamma) - \eta\mu(A-B+\Gamma) +$$

$$+\eta\mu(B-\Gamma-A) - \eta\mu(B-\Gamma+A) + \eta\mu(\Gamma+A-B) - \eta\mu(\Gamma-A+B)]$$

$$= \frac{1}{8}[\eta\mu(3B+\Gamma-A) - \eta\mu(3A-B+\Gamma) +$$

$$+\eta\mu(3\Gamma+A-B) - \eta\mu(3B-\Gamma+A) + \eta\mu(3A+B-\Gamma) - \eta\mu(3\Gamma-A+B)]$$

$$= \frac{1}{8}[2\eta\mu(2B-2A)\sigma\upsilon\nu(A+B+\Gamma) +$$

$$+2\eta\mu(2\Gamma-2B)\sigma\upsilon\nu(A+B+\Gamma) + 2\eta\mu(2A-2B)\sigma\upsilon\nu(A+B+\Gamma)]$$

$$= \frac{1}{4}\sigma\upsilon\nu(A+B+\Gamma)[\eta\mu(2B-2A) + \eta\mu(2\Gamma-2B) + \eta\mu(2A-2\Gamma)]$$

$$= -\frac{1}{4}[2\eta\mu(\Gamma-A)\sigma\upsilon\nu(2B-A-\Gamma) - 2\eta\mu(\Gamma-A)\sigma\upsilon\nu(\Gamma-A)]$$

$$= -\frac{1}{2}\eta\mu(\Gamma-A)[\sigma\upsilon\nu(2B-A-\Gamma) - \sigma\upsilon\nu(\Gamma-A)] = -\eta\mu(\Gamma-A)\eta\mu(B-A)\eta\mu(\Gamma-B)$$

$$= -\eta\mu(B-\Gamma)\eta\mu(\Gamma-A)\eta\mu(A-B) = -\Pi\eta\mu(A-B)$$

Ἄρα : $\Sigma\, \sigma\upsilon\nu^3 A\eta\mu(B-\Gamma) + \Pi\eta\mu(A-B) = -\Pi\eta\mu(A-B) + \Pi\eta\mu(A-B) = 0.$

4. $\mathbf{\Sigma\, \sigma\upsilon\nu^3 A\sigma\upsilon\nu(B-\Gamma) + \Pi\sigma\upsilon\nu(A-B) - 3\Pi\sigma\upsilon\nu A - 1 = 0.}$

Λύσις. Ἐργαζόμενοι ὅπως προηγουμένως, εὑρίσκομεν εὐκόλως τὸ ζητούμενον.

5. $\mathbf{\Sigma\, \eta\mu 3A\sigma\upsilon\nu(B-\Gamma) = 0.}$

Λύσις. Ἐπειδὴ $\eta\mu 3A = 3\eta\mu A - 4\eta\mu^3 A$, ἕπεται ὅτι :

$$\eta\mu 3A\sigma\upsilon\nu(B-\Gamma) = (3\eta\mu A - 4\eta\mu^3 A)\sigma\upsilon\nu(B-\Gamma) =$$

$$= 3\eta\mu A\sigma\upsilon\nu(B-\Gamma) - 4\eta\mu^3 A\sigma\upsilon\nu(B-\Gamma)$$

Κατ' ἀκολουθίαν :

$$\Sigma\, \eta\mu 3A\sigma\upsilon\nu(B-\Gamma) = 3\Sigma\, \eta\mu A\sigma\upsilon\nu(B-\Gamma) - 4\Sigma\, \eta\mu^3 A\sigma\upsilon\nu(B-\Gamma) =$$

$$= 3\cdot 4\eta\mu A\eta\mu B\eta\mu\Gamma - 4\cdot 3\eta\mu A\eta\mu B\eta\mu\Gamma = 0,$$ (ἄσκ. 95,3 καὶ 96,2).

97. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. Σ **συν3Αημ(Β—Γ)+4Πημ(Α—Β)=0.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$συν3Αημ(Β-Γ)=\frac{1}{2}\cdot 2συν3Αημ(Β-Γ)=\frac{1}{2}[ημ(3Α+Β-Γ)-ημ(3Α-Β+Γ)]$$

καὶ κατ' ἀκολουθίαν :

$$\Sigma\, συν3Αημ(Β-Γ)=\frac{1}{2}[ημ(3Α+Β-Γ)-ημ(3Α-Β+Γ)+ημ(3Β+Γ-Α)-$$
$$-ημ(3Β-Γ+Α)+ημ(3Γ+Α-Β)-ημ(3Γ-Α+Β)]$$
$$=4συν(Α+Β+Γ)Πημ(Β-Γ)=-4Πημ(Β-Γ), \quad (ἄσκ.\ 96,3)$$

Ἄρα : $\Sigma συν3Αημ(Β-Γ)+4Πημ(Β-Γ)=-4Πημ(Β-Γ)+4Πημ(Β-Γ)=0.$

2. Σ **συν3Ασυν(Β—Γ)+4Πσυν(Α—Β)=0.**

Λύσις. Ἔχομεν διαδοχικῶς :

$$συν3Ασυν(Β-Γ)=\frac{1}{2}\cdot 2συν3Ασυν(Β-Γ)=$$
$$=\frac{1}{2}[συν(3Α+Β-Γ)+συν(3Α-Β+Γ)].$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν γραμμάτων Α, Β, Γ, ἔχομεν :

$$\Sigma\, συν3Ασυν(Β-Γ)=\frac{1}{2}[συν(3Α+Β-Γ)+συν(3Α-Β+Γ)+συν(3Β+Γ-Α)+$$
$$+συν(3Β-Γ+Α)+συν(3Γ+Α-Β)+συν(3Γ-Α+Β)]$$
$$=-4Πσυν(Α-Β).$$

Κατ' ἀκολουθίαν :

$$\Sigma\, συν3Ασυν(Β-Γ)+4Πσυν(Α-Β)=-4Πσυν(Α-Β)+4Πσυν(Α-Β)=0.$$

3. Σ **ημ3Αημ³(Β—Γ)=0.**

Λύσις. Ἔχομεν : $ημ^3(Β-Γ)=\frac{1}{4}[3ημ(Β-Γ)-ημ(3Β-3Γ)]$

καὶ κατ' ἀκολουθίαν :

$$ημ3Αημ^3(Β-Γ)=\frac{1}{4}ημ3Α[3ημ(Β-Γ)-ημ(3Β-3Γ)]=$$
$$=\frac{3}{4}ημ3Αημ(Β-Γ)-\frac{1}{4}ημ3Αημ(3Β-3Γ)$$
$$=\frac{3}{8}\cdot 2ημ3Αημ(Β-Γ)-\frac{1}{8}\cdot 2ημ3Αημ(3Β-3Γ)$$
$$=\frac{3}{8}[συν(3Α-Β+Γ)-συν(3Α+Β-Γ)]-\frac{1}{8}[συν(3Α-3Β+3Γ)-$$
$$-συν(3Α+3Β-3Γ)].$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν Α, Β, Γ, λαμβάνομεν:

$$\Sigma\, \text{ημ}3\text{Α}\,\text{ημ}^3(\text{Β}-\Gamma)= \frac{3}{8}\left[\begin{array}{l}\text{συν}(3\text{Α}-\text{Β}+\Gamma)-\text{συν}(3\text{Α}+\text{Β}-\Gamma)+\\ \text{συν}(3\text{Β}-\Gamma+\text{Α})-\text{συν}(3\text{Β}+\Gamma-\text{Α})+\\ \text{συν}(3\Gamma-\text{Α}+\text{Β})-\text{συν}(3\Gamma+\text{Α}-\text{Β})-\end{array}\right]-$$

$$-\frac{1}{8}\left[\begin{array}{l}\text{συν}(3\text{Α}-3\text{Β}+3\Gamma)-\text{συν}(3\text{Α}+3\text{Β}-3\Gamma)+\\ \text{συν}(3\text{Β}-3\Gamma+3\text{Α})-\text{συν}(3\text{Β}+3\Gamma-3\text{Α})+\\ \text{συν}(3\Gamma-3\text{Α}+3\text{Β})-\text{συν}(3\Gamma+3\text{Α}-3\text{Β})\end{array}\right]$$

$$=\frac{3}{8}[\text{συν}(3\text{Α}-\text{Β}+\Gamma)-\text{συν}(3\text{Β}+\Gamma-\text{Α})+\text{συν}(3\text{Β}-\Gamma+\text{Α})-\text{συν}(3\Gamma+\text{Α}-\text{Β})+$$

$$+\text{συν}(3\Gamma-\text{Α}+\text{Β})-\text{συν}(3\text{Α}+\text{Β}-\Gamma)]=$$

$$=\frac{3}{8}[2\text{ημ}(\text{Α}+\text{Β}+\Gamma)\text{ημ}(\text{Β}-\text{Α})+2\text{ημ}(\text{Α}+\text{Β}+\Gamma)\text{ημ}(\Gamma-\text{Β})+$$

$$+2\text{ημ}(\text{Α}+\text{Β}+\Gamma)\text{ημ}(\text{Α}-\Gamma)]=$$

$$=\frac{3}{4}\text{ημ}(\text{Α}+\text{Β}+\Gamma)[\text{ημ}(\text{Β}-\text{Α})+\text{ημ}(\Gamma-\text{Β})+\text{ημ}(\text{Α}-\Gamma)]=$$

$$=\frac{3}{4}\cdot 0\cdot[\text{ημ}(\text{Β}-\text{Α})+\text{ημ}(\Gamma-\text{Β})+\text{ημ}(\text{Α}-\Gamma)]=0.$$

4. $\boldsymbol{\Sigma\, \text{ημ}3\text{Α}\,\text{συν}^3(\text{Β}-\Gamma)-\Pi\,\text{ημ}3\text{Α}=0.}$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\text{ημ}3\text{Α}\,\text{συν}^3(\text{Β}-\Gamma)=(\text{ημ}3\text{Α})\cdot\frac{1}{4}[\text{συν}(3\text{Β}-3\Gamma)+3\text{συν}(\text{Β}-\Gamma)]=$$

$$=\frac{1}{4}\text{ημ}3\text{Α}\,\text{συν}(3\text{Β}-3\Gamma)+\frac{3}{4}\text{ημ}3\text{Α}\,\text{συν}(\text{Β}-\Gamma)=$$

$$=\frac{1}{8}\cdot 2\text{ημ}3\text{Α}\,\text{συν}(3\text{Β}-3\Gamma)+\frac{3}{8}\cdot 2\text{ημ}3\text{Α}\,\text{συν}(\text{Β}-\Gamma)=$$

$$=\frac{1}{8}[\text{ημ}(3\text{Α}+3\text{Β}-3\Gamma)+\text{ημ}(3\text{Α}-3\text{Β}+3\Gamma)]+\frac{3}{8}[\text{ημ}(3\text{Α}+\text{Β}-\Gamma)+$$

$$+\text{ημ}(3\text{Α}-\text{Β}+\Gamma)] \qquad (1)$$

Ἐπειδὴ δέ:

$3\text{Α}+3\text{Β}-3\Gamma=3(\text{Α}+\text{Β})-3\Gamma=3(\pi-\Gamma)-3\Gamma=3\pi-6\Gamma=2\pi+(\pi-6\Gamma)$, ἕπεται ὅτι:

$$\text{ημ}(3\text{Α}+3\text{Β}-3\Gamma)=\text{ημ}[2\pi+(\pi-6\Gamma)]=\text{ημ}(\pi-6\Gamma)=\text{ημ}6\Gamma$$

καὶ $3\text{Α}+\text{Β}-\Gamma=2\text{Α}+(\text{Α}+\text{Β})-\Gamma=2\text{Α}+(\pi-\Gamma)-\Gamma=2\text{Α}+\pi-2\Gamma=\pi+(2\text{Α}-2\Gamma)$,

ἡ (1) γράφεται:

$$\text{ημ}3\text{Α}\,\text{συν}^3(\text{Β}-\Gamma)=\frac{1}{8}[\text{ημ}6\Gamma+\text{ημ}6\text{Β}]+\frac{3}{8}[\text{ημ}(2\Gamma-2\text{Α})+\text{ημ}(2\pi-2\text{Α})].$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν γραμμάτων Α, Β, Γ, λαμβάνομεν:

$$\Sigma\, \text{ημ}3\text{Α}\,\text{συν}^3(\text{Β}-\Gamma)=\frac{1}{8}\left[\begin{array}{c}\text{ημ}6\Gamma+\text{ημ}6\text{Β}+\\ \text{ημ}6\text{Α}+\text{ημ}6\Gamma+\\ \text{ημ}6\text{Β}+\text{ημ}6\text{Α}\end{array}\right]+\frac{3}{8}$$

$$\left[\begin{array}{l}\text{ημ}(2\Gamma-2\text{Α})+\text{ημ}(2\text{Β}-2\text{Α})+\\ \text{ημ}(2\text{Α}-2\text{Β})+\text{ημ}(2\Gamma-2\text{Β})+\\ \text{ημ}(2\text{Β}-2\Gamma)+\text{ημ}(2\text{Α}-2\Gamma)\end{array}\right]=$$

$$=\frac{1}{4}[\eta\mu 6A+\eta\mu 6B+\eta\mu 6\Gamma]=\frac{1}{4}[2\eta\mu(3A+3B)\sigma\upsilon\nu(3A-3B)+2\eta\mu 3\Gamma\sigma\upsilon\nu 3\Gamma]=$$

$$=\frac{1}{2}[\eta\mu(3A+3B)\sigma\upsilon\nu(3A-3B)+\eta\mu 3\Gamma\sigma\upsilon\nu 3\Gamma]=$$

$$=\frac{1}{2}[\eta\mu 3\Gamma\sigma\upsilon\nu(3A-3B)+\eta\mu 3\Gamma\sigma\upsilon\nu 3\Gamma]=$$

$$=\frac{1}{2}\eta\mu 3\Gamma[\sigma\upsilon\nu(3A-3B)+\sigma\upsilon\nu 3\Gamma]=\frac{1}{2}\eta\mu 3\Gamma[\sigma\upsilon\nu(3A-3B)-\sigma\upsilon\nu(3A+3B)]=$$

$$=\eta\mu 3A\eta\mu 3B\eta\mu 3\Gamma=\Pi\eta\mu 3A.$$ Ἄρα :

$$\Sigma\,\eta\mu 3A\sigma\upsilon\nu^3(B-\Gamma)-\Pi\eta\mu 3A=\Pi\eta\mu 3A-\Pi\eta\mu 3A=0.$$

5. $$\boldsymbol{\Sigma\,\sigma\upsilon\nu 3A\eta\mu^3(B-\Gamma)+3\Pi\eta\mu(A-B)=0.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu 3A\eta\mu^3(B-\Gamma)=(\sigma\upsilon\nu 3A)\cdot\frac{1}{4}[3\eta\mu(B-\Gamma)-\eta\mu(3B-3\Gamma)]=$$

$$=\frac{3}{4}\sigma\upsilon\nu 3A\eta\mu(B-\Gamma)-\frac{1}{4}\sigma\upsilon\nu 3A\eta\mu(3B-3\Gamma)=$$

$$=\frac{3}{8}\cdot 2\sigma\upsilon\nu 3A\eta\mu(B-\Gamma)-\frac{1}{8}\cdot 2\sigma\upsilon\nu 3A\eta\mu(3B-3\Gamma)=$$

$$=\frac{3}{8}[\eta\mu(3A+B-\Gamma)-\eta\mu(3A-B+\Gamma)]-\frac{1}{8}[\eta\mu(3A+3B-3\Gamma)-\eta\mu(3A-3B+3\Gamma)]$$

$$=\frac{3}{8}[\eta\mu(2\Gamma-2A)-\eta\mu(2B-2A)]-\frac{1}{8}[\eta\mu 6\Gamma-\eta\mu 6B].$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν A, B, Γ, λαμβάνομεν:

$$\Sigma\sigma\upsilon\nu 3A\eta\mu^3(B-\Gamma)=\frac{3}{8}\begin{bmatrix}\eta\mu(2\Gamma-2A)-\eta\mu(2B-2A)+\\ \eta\mu(2A-2B)-\eta\mu(2\Gamma-2B)+\\ \eta\mu(2B-2\Gamma)-\eta\mu(2A-2\Gamma)\end{bmatrix}-\frac{1}{8}\begin{bmatrix}\eta\mu 6\Gamma-\eta\mu 6B+\\ \eta\mu 6A-\eta\mu 6\Gamma+\\ \eta\mu 6B-\eta\mu 6A\end{bmatrix}=$$

$$=\frac{3}{8}[2\eta\mu(2A-2B)+2\eta\mu(2B-2\Gamma)+2\eta\mu(2\Gamma-2A)]=$$

$$=\frac{3}{4}[\eta\mu 2A-2B)+\eta\mu(2B-2\Gamma)+\eta\mu(2\Gamma-2A)]=$$

$$=\frac{3}{4}[2\eta\mu(A-\Gamma)\sigma\upsilon\nu(A-2B+\Gamma)+2\eta\mu(\Gamma-A)\sigma\upsilon\nu(\Gamma-A)]=$$

$$=\frac{3}{2}\eta\mu(A-\Gamma)[\sigma\upsilon\nu(A-2B+\Gamma)-\sigma\upsilon\nu(\Gamma-A)]=$$

$$=\frac{3}{2}\cdot\eta\mu(A-\Gamma)[2\eta\mu(\Gamma-B)\eta\mu(B-A)]=3\eta\mu(A-\Gamma)\eta\mu(\Gamma-B\eta\mu(B-A)=$$

$$=-3\eta\mu(A-B)\eta\mu(B-\Gamma)\eta\mu(\Gamma-A)=-3\Pi\eta\mu(A-B).$$

Κατ' ἀκολουθίαν:

$$\Sigma\sigma\upsilon\nu 3A\eta\mu^3(B-\Gamma)+3\Pi\eta\mu(A-B)=-3\Pi\eta\mu(A-B)+3\Pi\eta\mu(A-B)=0.$$

6. $\Sigma\ \eta\mu A \eta\mu^3(B-\Gamma) - 4\Pi\eta\mu A\eta\mu(B-\Gamma) = 0.$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu A\cdot\eta\mu^3(B-\Gamma) = (\eta\mu A)\cdot\frac{1}{4}[3\eta\mu(B-\Gamma)-\eta\mu(3B-3\Gamma)] =$$

$$=\frac{3}{4}\eta\mu A\eta\mu(B-\Gamma)-\frac{1}{4}\eta\mu A\eta\mu(3B-3\Gamma) =$$

$$=\frac{3}{4}\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)-\frac{1}{8}\cdot 2\eta\mu A\eta\mu(3B-3\Gamma) =$$

$$=\frac{3}{4}(\eta\mu^2 B-\eta\mu^2\Gamma)-\frac{1}{8}[\sigma\upsilon\nu(A-3B+3\Gamma)-\sigma\upsilon\nu(A+3B-3\Gamma)].$$

Διὰ κυκλικῆς ἐναλλαγῆς τῶν γραμμάτων Α, Β, Γ, ἔχομεν:

$$\Sigma\ \eta\mu A\eta\mu^3(B-\Gamma) = \frac{3}{4}\begin{bmatrix}\eta\mu^2B-\eta\mu^2\Gamma+\\ \eta\mu^2\Gamma-\eta\mu^2A+\\ \eta\mu^2A-\eta\mu^2B\end{bmatrix}-$$

$$-\frac{1}{8}\begin{bmatrix}\sigma\upsilon\nu(A-3B+3\Gamma)-\sigma\upsilon\nu(A+3B-3\Gamma)+\\ \sigma\upsilon\nu(B-3\Gamma+3A)-\sigma\upsilon\nu(B+3\Gamma-3A)+\\ \sigma\upsilon\nu(\Gamma-3A+3B)-\sigma\upsilon\nu(\Gamma+3A-3B)\end{bmatrix}=$$

$$=-\frac{1}{8}[\sigma\upsilon\nu(A-3B+3\Gamma)-\sigma\upsilon\nu(\Gamma+3A-3B)+$$

$$+\sigma\upsilon\nu(B-3\Gamma+3A)-\sigma\upsilon\nu(A+3B-3\Gamma)+\sigma\upsilon\nu(\Gamma-3A+3B)-\sigma\upsilon\nu(B+3\Gamma-3A)]$$

$$=-\frac{1}{8}[2\eta\mu(2A+2\Gamma-3B)\eta\mu(\Gamma-A)+2\eta\mu(2B+2A-3\Gamma)\eta\mu(A-B)+$$
$$+2\eta\mu(2\Gamma+2B-3A)\eta\mu(B-\Gamma)]$$

$$=-\frac{1}{4}[\eta\mu(2\pi-5B)\eta\mu(\Gamma-A)+\eta\mu(2\pi-5\Gamma)\eta\mu(A-B)+\eta\mu(2\pi-5A)\eta\mu(B-\Gamma)]$$

$$=-\frac{1}{4}[\eta\mu 5B\eta\mu(\Gamma-A)+\eta\mu 5\Gamma\eta\mu(A-B)+\eta\mu 5A\eta\mu(B-\Gamma)] \qquad (1)$$

Ἀντικαθιστῶντες τὰ ημ5Α, . . ., ημ(Γ—Α) . . ., διὰ τῶν ἴσων των καὶ ἐκτελοῦντες τὰς σημειωμένας πράξεις, εὑρίσκομεν:

$$\Sigma\cdot\eta\mu A\eta\mu^3(B-\Gamma) = 4\Pi\eta\mu A\eta\mu(B-\Gamma), \quad \text{ὁπότε:}$$

$$\Sigma\ \eta\mu A\eta\mu^3(B-\Gamma)-4\Pi\eta\mu A\eta\mu(B-\Gamma)=4\Pi\eta\mu A\eta\mu(B-\Gamma)-4\Pi\eta\mu A\eta\mu(B-\Gamma)=0.$$

Ὁμοίως ἀποδεικνύεται ὅτι:

8. $\Sigma\ \eta\mu^3 A\eta\mu^3(B-\Gamma)-3\Pi\eta\mu A\eta\mu(B-\Gamma)=0$

9. $\Sigma\ \eta\mu A\eta\mu 3(B-\Gamma)+16\Pi\eta\mu A\eta\mu(B-\Gamma)=0$

98. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

1α. $\Sigma\ \eta\mu(kA) = -4\eta\mu\frac{kA}{2}\eta\mu\frac{kB}{2}\eta\mu\frac{k\Gamma}{2}$, ἂν $k=4\mu,\ \mu\in N$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\Sigma\ \eta\mu(kA)=\eta\mu(kA)+\eta\mu(kB)+\eta\mu(k\Gamma)$$

$$=2\eta\mu\frac{k(A+B)}{2}\sigma\upsilon\nu\frac{k(A-B)}{2}+2\eta\mu\frac{k\Gamma}{2}\sigma\upsilon\nu\frac{k\Gamma}{2} \qquad (1)$$

Ἐὰν $k=4\mu$, τότε: $\eta\mu\,\frac{k(A+B)}{2}=\eta\mu\left(\frac{k\pi}{2}-\frac{k\Gamma}{2}\right)=-\eta\mu\,\frac{k\Gamma}{2}$

καὶ $\sigma\upsilon\nu\,\frac{k(A+B)}{2}=\sigma\upsilon\nu\left(\frac{k\pi}{2}-\frac{k\Gamma}{2}\right)=\sigma\upsilon\nu\,\frac{k\Gamma}{2}$, καὶ ἡ (1) γίνεται:

$$\Sigma\,\eta\mu(kA)=2\eta\mu\,\frac{k\Gamma}{2}\left[-\sigma\upsilon\nu\,\frac{k(A-B)}{2}+\sigma\upsilon\nu\,\frac{k(A+B)}{2}\right]=$$

$$=-4\eta\mu\,\frac{kA}{2}\,\eta\mu\,\frac{kB}{2}\,\eta\mu\,\frac{k\Gamma}{2}.$$

1β. $\boldsymbol{\Sigma\,\eta\mu(kA)=4\sigma\upsilon\nu\frac{kA}{2}\,\sigma\upsilon\nu\frac{kB}{2}\,\sigma\upsilon\nu\frac{k\Gamma}{2}}$, **ἂν $k=4\mu+1$.**

Λύσις. Διὰ $k=4\mu+1$, εἶναι:

$$\eta\mu\,\frac{k(A+B)}{2}=\sigma\upsilon\nu\,\frac{k\Gamma}{2}\quad\text{καὶ}\quad\sigma\upsilon\nu\,\frac{k(A+B)}{2}=\eta\mu\,\frac{k\Gamma}{2}.$$

Κατ' ἀκολουθίαν:

$$\Sigma\,\eta\mu(kA)=2\sigma\upsilon\nu\,\frac{k\Gamma}{2}\left[\sigma\upsilon\nu\,\frac{k(A-B)}{2}+\sigma\upsilon\nu\,\frac{k(A+B)}{2}\right]=$$

$$=4\sigma\upsilon\nu\,\frac{kA}{2}\,\sigma\upsilon\nu\,\frac{kB}{2}\,\sigma\upsilon\nu\,\frac{k\Gamma}{2}.$$

1γ. $\boldsymbol{\Sigma\,\eta\mu(kA)=4\eta\mu\,\frac{kA}{2}\,\eta\mu\frac{kB}{2}\,\eta\mu\frac{k\Gamma}{2}}$, **ἂν $k=4\mu+2$**

Λύσις. Διὰ $k=4\mu+2$ εἶναι:

$$\eta\mu\,\frac{k(A+B)}{2}=\eta\mu\,\frac{k\Gamma}{2}\quad\text{καὶ}\quad\sigma\upsilon\nu\,\frac{k(A+B)}{2}=-\sigma\upsilon\nu\,\frac{k\Gamma}{2}$$

καὶ κατ' ἀκολουθίαν:

$$\Sigma\,\eta\mu(kA)=2\eta\mu\,\frac{k\Gamma}{2}\left[\sigma\upsilon\nu\,\frac{k(A-B)}{2}-\sigma\upsilon\nu\,\frac{k(A+B)}{2}\right]=4\eta\mu\,\frac{kA}{2}\,\eta\mu\,\frac{kB}{2}\,\eta\mu\,\frac{k\Gamma}{2}.$$

1δ. $\boldsymbol{\Sigma\,\eta\mu(kA)=-4\sigma\upsilon\nu\frac{kA}{2}\,\sigma\upsilon\nu\frac{kB}{2}\,\sigma\upsilon\nu\frac{k\Gamma}{2}}$, **ἂν $k=4\mu+3$.**

Λύσις. Διὰ $k=4\mu+3$ εἶναι:

$$\eta\mu\,\frac{k(A+B)}{2}=-\sigma\upsilon\nu\,\frac{k\Gamma}{2}\quad\text{καὶ}\quad\sigma\upsilon\nu\,\frac{k(A+B)}{2}=-\eta\mu\,\frac{k\Gamma}{2},\quad\text{ὁπότε:}$$

$$\Sigma\,\eta\mu(kA)=2\sigma\upsilon\nu\,\frac{k\Gamma}{2}\left[-\sigma\upsilon\nu\,\frac{k(A-B)}{2}-\sigma\upsilon\nu\,\frac{k(A+B)}{2}\right]=$$

$$=-4\sigma\upsilon\nu\,\frac{kA}{2}\,\sigma\upsilon\nu\,\frac{kB}{2}\,\sigma\upsilon\nu\,\frac{k\Gamma}{2}.$$

2α. Σ **συν(kA)=−1+4συν** $\frac{kA}{2}$ **συν** $\frac{kB}{2}$ **συν** $\frac{k\Gamma}{2}$, ἂν k=4μ.

Λύσις. Ἐὰν k=4μ, θὰ εἶναι:

$$ημ\,\frac{k(A+B)}{2}=-ημ\,\frac{k\Gamma}{2},\quad καὶ\quad συν\,\frac{k(A+B)}{2}=συν\,\frac{k\Gamma}{2}$$

καὶ κατ' ἀκολουθίαν:

$$\Sigma\,συν(kA)=συν(kA)+συν(kB)+συν(k\Gamma)=$$

$$=2συν\,\frac{k(A+B)}{2}\,συν\,\frac{k(A-B)}{2}+2συν^2\,\frac{k\Gamma}{2}-1=$$

$$=-1+2συν\,\frac{k\Gamma}{2}\left[συν\,\frac{k(A-B)}{2}+συν\,\frac{k(A+B)}{2}\right]=$$

$$=-1+4συν\,\frac{kA}{2}\,συν\,\frac{kB}{2}\,συν\,\frac{k\Gamma}{2}.$$

2β. Σ **συν(kA)=1+4ημ** $\frac{kA}{2}$ **ημ** $\frac{kB}{2}$ **ημ** $\frac{k\Gamma}{2}$, ἂν k=4μ+1.

Λύσις. Διὰ k=4μ+1, εἶναι:

$$ημ\,\frac{k(A+B)}{2}=συν\,\frac{k\Gamma}{2}\quad καὶ\quad συν\,\frac{k(A+B)}{2}=ημ\,\frac{k\Gamma}{2},\quad ὁπότε:$$

$$\Sigma\,συν(kA)=συν(kA)+συν(kB)+συν(k\Gamma)=$$

$$=2συν\,\frac{k(A+B)}{2}\,συν\,\frac{k(A-B)}{2}+1-2ημ^2\,\frac{k\Gamma}{2}=$$

$$=1+2ημ\,\frac{k\Gamma}{2}\left[συν\,\frac{k(A-B)}{2}-συν\,\frac{k(A+B)}{2}\right]=1+4ημ\,\frac{kA}{2}\,ημ\,\frac{kB}{2}\,ημ\,\frac{k\Gamma}{2}.$$

2γ. Σ **συν(kA)=−1−4συν** $\frac{kA}{2}$ **συν** $\frac{kB}{2}$ **συν** $\frac{k\Gamma}{2}$, ἂν k=4μ+2

Λύσις. Διὰ k=4μ+2, εἶναι:

$$ημ\,\frac{k(A+B)}{2}=ημ\,\frac{k\Gamma}{2}\quad καὶ\quad συν\,\frac{k(A+B)}{2}=-συν\,\frac{k\Gamma}{2},\quad ὁπότε:$$

$$\Sigma\,συν(kA)=συν(kA)+συν(kB)+συν(k\Gamma)=$$

$$=2συν\,\frac{k(A+B)}{2}\,συν\,\frac{k(A-B)}{2}+2συν^2\,\frac{k\Gamma}{2}-1=$$

$$=-1+2συν\,\frac{k\Gamma}{2}\left[-συν\,\frac{k(A-B)}{2}-συν\,\frac{k(A+B)}{2}\right]=$$

$$=-1-4συν\,\frac{kA}{2}\,συν\,\frac{kB}{2}\,συν\,\frac{k\Gamma}{2}.$$

2δ. $\Sigma\, \sigma\upsilon\nu(kA) = 1 - 4\eta\mu\frac{kA}{2}\,\eta\mu\frac{kB}{2}\,\eta\mu\frac{k\Gamma}{2}$, ἂν $k = 4\mu + 3$.

Λύσις. Διὰ $k = 4\mu + 3$, εἶναι :

$$\eta\mu\,\frac{k(A+B)}{2} = -\sigma\upsilon\nu\,\frac{k\Gamma}{2} \quad \text{καὶ} \quad \sigma\upsilon\nu\frac{k(A+B)}{2} = -\eta\mu\,\frac{k\Gamma}{2}, \text{ ὁπότε:}$$

$$\Sigma\,\sigma\upsilon\nu(kA) = \sigma\upsilon\nu(kA) + \sigma\upsilon\nu(kB) + \sigma\upsilon\nu(k\Gamma) =$$

$$= 2\sigma\upsilon\nu\,\frac{k(A+B)}{2}\,\sigma\upsilon\nu\,\frac{k(A-B)}{2} + 1 - 2\eta\mu^2\,\frac{k\Gamma}{2} =$$

$$= 1 + 2\eta\mu\frac{k\Gamma}{2}\left[-\sigma\upsilon\nu\,\frac{k(A-B)}{2} + \sigma\upsilon\nu\,\frac{k(A+B)}{2}\right) =$$

$$= 1 - 4\eta\mu\,\frac{kA}{2}\,\eta\mu\,\frac{kB}{2}\,\eta\mu\,\frac{k\Gamma}{}.$$

3. $\Sigma\sigma\upsilon\nu^2(kA) = 1 + 2(-1)^k\,\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma),\ k \in Z.$

Λύσις. Ἔχομεν :

$$\sigma\upsilon\nu(k\Gamma) = \sigma\upsilon\nu[k\pi - k(A+B)] = \pm\sigma\upsilon\nu k(A+B) = \pm[\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB) - \eta\mu(kA)\eta\mu(kB)]$$

καὶ κατ' ἀκολουθιαν :

$$\sigma\upsilon\nu^2(k\Gamma) = \sigma\upsilon\nu^2(kA)\sigma\upsilon\nu^2(kB) + \eta\mu^2(kA)\eta\mu^2(kB) - 2\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\eta\mu(kA)\eta\mu(kB)$$

$$= \sigma\upsilon\nu^2(kA)\sigma\upsilon\nu^2(kB) + [1 - \sigma\upsilon\nu^2(kA)][1 - \sigma\upsilon\nu^2(kB)] -$$

$$-2\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\eta\mu(kA)\eta\mu(kB) =$$

$$= 1 - \sigma\upsilon\nu^2(kA) - \sigma\upsilon\nu^2(kB) + 2\sigma\upsilon\nu^2(kA)\sigma\upsilon\nu^2(kB) -$$

$$-2\sigma\upsilon\nu(kA)\sigma\upsilon\nu(k(B\eta\mu(kA)\eta\mu(kB)$$

ὁπότε $\sigma\upsilon\nu^2(kA) + \sigma\upsilon\nu^2(kB) + \sigma\upsilon\nu^2(k\Gamma) =$

$$= 2\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)[\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB) - \eta\mu(kA)\eta\mu(kB)] + 1$$

$$= 1 + 2\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu k(A+B)$$

$$= 1 + 2\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu[k\pi - k\Gamma] \qquad (1)$$

Ἐὰν $k =$ ἄρτιος, τότε: $\sigma\upsilon\nu(k\pi - k\Gamma) = \sigma\upsilon\nu(k\Gamma)$

καὶ ἐὰν $k =$ περιττός, τότε: $\sigma\upsilon\nu(k\pi - k\Gamma) = -\sigma\upsilon\nu(k\Gamma)$, καὶ ἡ (1) γίνεται :

$$\sigma\upsilon\nu^2(kA) + \sigma\upsilon\nu^2(kB) + \sigma\upsilon\nu^2(k\Gamma) = 1 + 2(-1)^k\,\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma)$$

4. $\Sigma\,\eta\mu^2(kB) = 2 - 2(-1)^k\,\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma),\ k \in Z.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\Sigma\,\eta\mu^2(kA) = \Sigma\,[1 - \sigma\upsilon\nu^2(kA)] = 3 - \Sigma\,\sigma\upsilon\nu^2(kA) =$$

$$= 3 - [2 - 2(-1)^k\,\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma)]$$

$$= 1 + 2(-1)^k\,\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma).$$

99. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:

1. $$\Sigma\, ημ(B+2Γ)=4ημ\frac{B-Γ}{2}\,ημ\frac{Γ-A}{2}\,ημ\frac{A-B}{2}.$$

Λύσις. Ἔχομεν:

$$ημ(B+2Γ)=ημ(π+Γ-A)=-ημ(Γ-A)=ημ(A-Γ),$$

Ἄρα, διὰ κυκλικῆς ἐναλλαγῆς τῶν Α, Β, Γ, θὰ ἔχωμεν διαδοχικῶς:

$$\Sigma\, ημ(B+2Γ)=ημ(B+2Γ)+ημ(Γ+2A)+ημ(A+2B)$$
$$=ημ(A-Γ)+ημ(B-A)+ημ(Γ-B)$$
$$=2ημ\frac{B-Γ}{2}\,συν\frac{-Γ-B+2A}{2}+2ημ\frac{Γ-B}{2}\,συν\frac{Γ-B}{2}$$
$$=2ημ\frac{B-Γ}{2}\left[συν\frac{-Γ-B+2A}{2}-συν\frac{Γ-B}{2}\right]$$
$$=2ημ\frac{B-Γ}{2}\cdot 2ημ\frac{A-B}{2}\,ημ\frac{Γ-A}{2}$$
$$=4ημ\frac{B-Γ}{2}\,ημ\frac{Γ-A}{2}\,ημ\frac{A-B}{2}$$

2. $$\Sigma\, ημ^4A=\frac{3}{2}+2συνAσυνBσυνΓ+\frac{1}{2}συν2Aσυν2Bσυν2Γ.$$

Λύσις. Ἐπειδή:

$$συν4A=1-2ημ^2 2A=1-8ημ^2Aσυν^2A=1-8ημ^2A+8ημ^4A,$$

ἕπεται ὅτι: $$ημ^4A=\frac{1}{8}συν4A+ημ^2A-\frac{1}{8}.$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν Α, Β, Γ, καὶ εἶτα διὰ προσθέσεως, λαμβάνομεν: $ημ^4A+ημ^4B+ημ^4Γ=$

$$=\frac{1}{8}(συν4A+συν4B+συν4Γ)+(ημ^2A+ημ^2B+ημ^2Γ)-\frac{3}{8} \qquad (1)$$

Ἔχοντες δ' ὑπ' ὄψει τὰς ἀσκήσεις (91, 33) καὶ ὅτι:

$\Sigma\, ημ^2A=2+2συνAσυνBσυνΓ)$, ἡ (1) γίνεται:

$$\Sigma\, ημ^4A=\frac{1}{8}[4συν2Aσυν2Bσυν2Γ-1]+(2+2συνAσυνBσυνΓ)-\frac{3}{8}$$
$$=\frac{1}{2}συν2Aσυν2Bσυν2Γ-\frac{1}{8}+2συνAσυνBσυνΓ+2-\frac{3}{8}.$$
$$=\frac{3}{2}+2συνAσυνBσυνΓ+\frac{1}{2}συν2Aσυν2Bσυν2Γ.$$

3. $$\Sigma\, συν^4A=\frac{1}{2}-2συνAσυνBσυνΓ+\frac{1}{2}συν2Aσυν2Bσυν2Γ.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$συν4A=1-2ημ^2 2A=1-8ημ^2Aσυν^2A=1-8συν^2A+8συν^4A,$$

ἐξ οὗ: $$\sigma\upsilon\nu^4 A = \frac{1}{8}\ \sigma\upsilon\nu 4A + \sigma\upsilon\nu^2 A - \frac{1}{8}.$$

Διὰ κυκλικῆς ἐναλλαγῆς τῶν Α, Β, Γ καὶ προσθέσεως, λαμβάνομεν:

$$\sigma\upsilon\nu^4 A + \sigma\upsilon\nu^4 B + \sigma\upsilon\nu^4 \Gamma = \frac{1}{8}\ (\sigma\upsilon\nu 4A + \sigma\upsilon\nu 4B + \sigma\upsilon\nu 4\Gamma) +$$

$$+ (\sigma\upsilon\nu^2 A + \sigma\upsilon\nu^2 B + \sigma\upsilon\nu^2 \Gamma) - \frac{3}{8} =$$

$$= \frac{1}{8}\ (4\sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma - 1) + 1 - 2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu \Gamma - \frac{3}{8} =$$

$$= \frac{1}{2}\ \sigma\upsilon\nu 2A\sigma\upsilon\nu 2B\sigma\upsilon\nu 2\Gamma - 2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu \Gamma + \frac{1}{2}.$$

4. $\mathbf{\Sigma\ \varepsilon\varphi(kA)\varepsilon\varphi(kB) = 1 - (-1)^k\ \tau\varepsilon\mu(kA)\tau\varepsilon\mu(kB)\tau\varepsilon\mu(k\Gamma)}$

Λύσις. Ἔχομεν; $A + B + \Gamma = \pi$ καὶ $kA + kB + k\Gamma = k\pi$.

Ἄρα: $\sigma\upsilon\nu(kA + kB + k\Gamma) = \sigma\upsilon\nu(k\pi) = (-1)^k$, ἤ

$$\sigma\upsilon\nu(kA + kB + k\Gamma) = \sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma) - \eta\mu(kA)\eta\mu(kB)\sigma\upsilon\nu(k\Gamma) -$$

$$- \eta\mu(kB)\eta\mu(k\Gamma)\sigma\upsilon\nu(kA) - \eta\mu(k\Gamma)\eta\mu(kA)\sigma\upsilon\nu(kB).$$

Εἶναι δέ:

$$\Sigma\ \varepsilon\varphi(kA)\varepsilon\varphi(kB) = \Sigma\ \frac{\eta\mu(kA)\eta\mu(kB)}{\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)} =$$

$$= \frac{\eta\mu(kA)\eta\mu(kB)}{\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)} + \frac{\eta\mu(kB)\eta\mu(k\Gamma)}{\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma)} + \frac{\eta\mu(k\Gamma)\eta\mu(kA)}{\sigma\upsilon\nu(k\Gamma)\sigma\upsilon\nu(kA)} =$$

$$= \frac{\eta\mu(kA)\eta\mu(kB)\sigma\upsilon\nu(k\Gamma) + \eta\mu(kB)\eta\mu(k\Gamma)\sigma\upsilon\nu(kA) + \eta\mu(k\Gamma)\eta\mu(kA)\sigma\upsilon\nu(kB)}{\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma)} =$$

$$= \frac{\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma) - \sigma\upsilon\nu(kA + kB + k\Gamma)}{\sigma\upsilon\nu(kA)\sigma\upsilon\nu(kB)\sigma\upsilon\nu(k\Gamma)} =$$

$$= 1 - (-1)^k\ \tau\varepsilon\mu(kA)\tau\varepsilon\mu(kB)\tau\varepsilon\mu(k\Gamma).$$

100. Εἰς πᾶν κυρτὸν τετράπλευρον ΑΒΓΔ νὰ ἀποδειχθῇ ὅτι:

1. $$\mathbf{\eta\mu A + \eta\mu B + \eta\mu\Gamma + \eta\mu\Delta = 4\eta\mu\ \frac{A+B}{2}\ \eta\mu\frac{B+\Gamma}{2}\ \eta\mu\frac{\Gamma+A}{2}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu A + \eta\mu B + \eta\mu\Gamma + \eta\mu\Delta = 2\eta\mu\frac{A+B}{2}\ \sigma\upsilon\nu\ \frac{A-B}{2} + 2\eta\mu\ \frac{\Gamma+\Delta}{2}\ \sigma\upsilon\nu\ \frac{\Gamma-\Delta}{2} =$$

$$= 2\eta\mu\ \frac{A+B}{2}\ \sigma\upsilon\nu\ \frac{A-B}{2} + 2\eta\mu\ \frac{A+B}{2}\ \sigma\upsilon\nu\ \frac{\Gamma-\Delta}{2} =$$

$$= 2\eta\mu\ \frac{A+B}{2}\left[\sigma\upsilon\nu\ \frac{A-B}{2} + \sigma\upsilon\nu\ \frac{\Gamma-\Delta}{2}\right]$$

$$= 4\eta\mu\,\frac{A+B}{2}\cdot\sigma\upsilon\nu\,\frac{A+\Gamma-B-\Delta}{4}\,\sigma\upsilon\nu\,\frac{A+\Delta-B-\Gamma}{4}$$

$$= 4\eta\mu\,\frac{A+B}{2}\cdot\eta\mu\,\frac{A+\Gamma}{2}\,\eta\mu\,\frac{B+\Gamma}{2}$$

καθόσον εἶναι:

$$\sigma\upsilon\nu\,\frac{A+\Gamma-B-\Delta}{4}=\eta\mu\,\frac{A+\Gamma}{2}\quad \text{καὶ}\quad \sigma\upsilon\nu\,\frac{A+\Delta-B-\Gamma}{4}=\eta\mu\,\frac{B+\Gamma}{2}.$$

2. $\mathbf{\eta\mu A-\eta\mu B+\eta\mu\Gamma-\eta\mu\Delta=4\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\frac{B+\Gamma}{2}\eta\mu\,\frac{\Gamma+A}{2}}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu A-\eta\mu B+\eta\mu\Gamma-\eta\mu\Delta=2\eta\mu\,\frac{A-B}{2}\,\sigma\upsilon\nu\,\frac{A+B}{2}+2\eta\mu\,\frac{\Gamma-\Delta}{2}\,\sigma\upsilon\nu\,\frac{\Gamma+\Delta}{2}=$$

$$= 2\eta\mu\,\frac{A-B}{2}\,\sigma\upsilon\nu\,\frac{A+B}{2}-2\eta\mu\,\frac{\Gamma-\Delta}{2}\,\sigma\upsilon\nu\,\frac{A+B}{2}$$

$$= 2\sigma\upsilon\nu\,\frac{A+B}{2}\left[\eta\mu\,\frac{A-B}{2}-\eta\mu\,\frac{\Gamma-\Delta}{2}\right]$$

$$= 4\sigma\upsilon\nu\,\frac{A+B}{2}\cdot\eta\mu\,\frac{A+\Delta-B-\Gamma}{4}\,\sigma\upsilon\nu\,\frac{A+\Gamma-B-\Delta}{4}$$

$$= 4\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\,\frac{B+\Gamma}{2}\,\eta\mu\,\frac{A+\Gamma}{2}.$$

3. $\mathbf{\sigma\upsilon\nu A+\sigma\upsilon\nu B+\sigma\upsilon\nu\Gamma+\sigma\upsilon\nu\Delta=4\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\frac{B+\Gamma}{2}\,\sigma\upsilon\nu\frac{\Gamma+A}{2}}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\sigma\upsilon\nu A+\sigma\upsilon\nu B+\sigma\upsilon\nu\Gamma+\sigma\upsilon\nu\Delta=2\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\,\frac{A-B}{2}+2\sigma\upsilon\nu\,\frac{\Gamma+\Delta}{2}\,\sigma\upsilon\nu\frac{\Gamma-\Delta}{2}=$$

$$= 2\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\,\frac{A-B}{2}-2\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\,\frac{\Gamma-\Delta}{2}$$

$$= 2\sigma\upsilon\nu\,\frac{A+B}{2}\left[\sigma\upsilon\nu\,\frac{A-B}{2}-\sigma\upsilon\nu\,\frac{\Gamma-\Delta}{2}\right]$$

$$= 4\sigma\upsilon\nu\,\frac{A+B}{2}\,\eta\mu\,\frac{A+\Gamma-B-\Delta}{4}\,\eta\mu\,\frac{B+\Gamma-A-\Delta}{4}$$

$$= 4\sigma\upsilon\nu\,\frac{A+B}{2}\,\sigma\upsilon\nu\,\frac{B+\Gamma}{2}\,\sigma\upsilon\nu\,\frac{A+\Gamma}{2},$$

καθόσον εἶναι:

$$\eta\mu\,\frac{A+\Gamma-B-\Delta}{4}=\eta\mu\,\frac{2A+2\Gamma-2\pi}{4}=\eta\mu\left(\frac{A+\Gamma}{2}-90^{0}\right)=-\sigma\upsilon\nu\,\frac{A+\Gamma}{2}$$

καὶ

$$\eta\mu\,\frac{B+\Gamma-A-\Delta}{4}=-\sigma\upsilon\nu\,\frac{B+\Gamma}{2}.$$

4. $\mathbf{συνA - συνB + συνΓ - συνΔ = 4ημ\frac{A+B}{2}\,ημ\frac{B+Γ}{2}\,συν\frac{Γ+A}{2}}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$συνA - συνB + συνΓ - συνΔ = 2ημ\frac{A+B}{2}\,ημ\frac{B-A}{2} + ημ\frac{Γ+Δ}{2}\,ημ\frac{Δ-Γ}{2} =$$

$$= 2ημ\frac{A+B}{2}\,ημ\frac{B-A}{2} + 2ημ\frac{A+B}{2}\,ημ\frac{Δ-Γ}{2}$$

$$= 2ημ\frac{A+B}{2}\left[ημ\frac{B-A}{2} + ημ\frac{Δ-Γ}{2}\right]$$

$$= 4ημ\frac{A+B}{2}\,ημ\frac{B+Δ-A-Γ}{4}\,συν\frac{B+Γ-A-Δ}{4}$$

$$= 4ημ\frac{A+B}{2}\,συν\frac{A+Γ}{2}\,ημ\frac{B+Γ}{2}$$

$$= 4ημ\frac{A+B}{2}\,ημ\frac{B+Γ}{2}\,συν\frac{Γ+A}{2}.$$

101. Ἐὰν εἰς τρίγωνον ΑΒΓ ἀληθεύουν αἱ ἰσότητες:

1) $\mathbf{σφ\frac{B}{2} = \frac{ημA+ημΓ}{ημB}}$, 2) $\mathbf{ημA = \frac{ημB+ημΓ}{συνB+συνΓ}}$

καὶ 3) **ημΓ=συνΑ+συνΒ, νὰ δειχθῇ ὅτι τὸ τρίγωνον τοῦτο εἶναι ὀρθογώνιον.**

Λύσις. Ἡ (1) γράφεται:

$$ημB\cdot σφ\frac{B}{2} = ημA + ημΓ = 2ημ\frac{A+Γ}{2}\,συν\frac{A-Γ}{2} = 2συν\frac{B}{2}\,συν\frac{A-Γ}{2}$$

ἢ
$$2ημ\frac{B}{2}\,συν\frac{B}{2}\cdot\frac{συν\frac{B}{2}}{ημ\frac{B}{2}} = 2συν\frac{B}{2}\,συν\frac{A-Γ}{2}$$

ἢ $συν\frac{B}{2} = συν\frac{A-Γ}{2}$. Ἄρα: $\frac{B}{2} = \frac{A-Γ}{2}$ ἢ $B+Γ=A \Rightarrow A=90^{\circ}$

ἢ $\frac{B}{2} = -\frac{A-Γ}{2} \Rightarrow Γ=A+B \Rightarrow Γ=90^{\circ}$.

Ἡ (2) γράφεται:

$$2ημ\frac{A}{2}\,συν\frac{A}{2} = \frac{2ημ\frac{B+Γ}{2}\,συν\frac{B-Γ}{2}}{2συν\frac{B+Γ}{2}\,συν\frac{B-Γ}{2}} = \frac{συν\frac{A}{2}}{ημ\frac{A}{2}}$$

ἐξ οὗ: $2ημ^{2}\frac{A}{2} = 1$ ἢ $ημ\frac{A}{2} = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2} = ημ45^{\circ}$.

Ἄρα: $\frac{A}{2} = 45^{\circ} \Rightarrow A=90^{\circ}$.

Ἡ (3) γράφεται :

$$2\eta\mu\frac{\Gamma}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}=2\sigma\upsilon\nu\frac{A+B}{2}\,\sigma\upsilon\nu\frac{A-B}{2}=2\eta\mu\frac{\Gamma}{2}\,\sigma\upsilon\nu\frac{A-B}{2}$$

ἢ $\sigma\upsilon\nu\frac{\Gamma}{2}=\sigma\upsilon\nu\frac{A-B}{2} \Longrightarrow \frac{\Gamma}{2}=\pm\frac{A-B}{2} \Longrightarrow \Gamma=\pm(A-B)$

ὅθεν ἢ $A=B+\Gamma \Longrightarrow A=90^0$

ἢ $B=A+\Gamma \Longrightarrow B=90^0.$

102. Ἐὰν αἱ γωνίαι τριγώνου ΑΒΓ ἐπαληθεύουν τὰς ἰσότητας:

1. $\varepsilon\varphi\frac{A}{2}+\varepsilon\varphi\frac{B}{2}+\varepsilon\varphi\frac{\Gamma}{2}+\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2}\,\varepsilon\varphi\frac{\Gamma}{2}=2.$

2: $\sigma\upsilon\nu^2A+\sigma\upsilon\nu^2B+\sigma\upsilon\nu^2\Gamma=1.$

3: $\eta\mu 2A+\eta\mu 2\Gamma=\eta\mu 2B.$

4: $\eta\mu 4A+\eta\mu 4B+\eta\mu 4\Gamma=0.$

νὰ ἀποδειχθῇ ὅτι τὸ τρίγωνον τοῦτο εἶναι ὀρθογώνιον.

Λύσις. Ἡ σχέσις γράφεται :

$$\varepsilon\varphi\frac{A}{2}+\varepsilon\varphi\frac{B}{2}+\varepsilon\varphi\frac{\Gamma}{2}-1-1+\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2}\,\varepsilon\varphi\frac{\Gamma}{2}=0$$

ἢ $$\varepsilon\varphi\frac{A}{2}\left(1-\varepsilon\varphi\frac{B}{2}\right)-\left(1-\varepsilon\varphi\frac{B}{2}\right)+$$

$$+\varepsilon\varphi\frac{\Gamma}{2}\left(1-\varepsilon\varphi\frac{B}{2}\right)-\left(1-\varepsilon\varphi\frac{B}{2}\right)\varepsilon\varphi\frac{\Gamma}{2}\,\varepsilon\varphi\frac{A}{2}=0$$

ἢ $$\left(1-\varepsilon\varphi\frac{B}{2}\right)\left(\varepsilon\varphi\frac{A}{2}-1+\varepsilon\varphi\frac{\Gamma}{2}-\varepsilon\varphi\frac{\Gamma}{2}\,\varepsilon\varphi\frac{A}{2}\right)=0$$

ἢ $$\left(1-\varepsilon\varphi\frac{B}{2}\right)\left(1-\varepsilon\varphi\frac{\Gamma}{2}\right)\left(1-\varepsilon\varphi\frac{A}{2}\right)=0$$

ὅθεν ἢ $1-\varepsilon\varphi\frac{B}{2}=0 \Longrightarrow \varepsilon\varphi\frac{B}{2}=1=\varepsilon\varphi 45^0 \Longrightarrow B=90^0,$

ἢ $1-\varepsilon\varphi\frac{\Gamma}{2}=0 \Longrightarrow \varepsilon\varphi\frac{\Gamma}{2}=1=\varepsilon\varphi 45^0 \Longrightarrow \Gamma=90^0,$

ἢ $1-\varepsilon\varphi\frac{A}{2}=0 \Longrightarrow \varepsilon\varphi\frac{A}{2}=1=\varepsilon\varphi 45^0 \Longrightarrow A=90^0,$

Λύσις. Ἡ (2) γράφεται ὡς ἑξῆς :

$$1-\eta\mu^2A+1-\eta\mu^2B+1-\eta\mu^2\Gamma=1$$

ἢ $$\eta\mu^2A+\eta\mu^2B+\eta\mu^2\Gamma=2$$

ἢ $$2+2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma=2$$

ἢ $$\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma=0$$

ὅθεν ἢ $A=90^0$ ἢ $B=90^0$ ἢ $\Gamma=90^0$.

Λύσις. Ἡ (3) γράφεται ὡς ἑξῆς:

$$\eta\mu 2A+\eta\mu 2\Gamma-\eta\mu 2B=0 \quad \text{ἢ} \quad 4\sigma\upsilon\nu A\sigma\upsilon\nu\Gamma\eta\mu B=0.$$

Ἐπειδὴ $\eta\mu B\neq 0$, ἕπεται: $\sigma\upsilon\nu A\sigma\upsilon\nu\Gamma=0$, ἐξ οὗ: ἢ $A=90^0$ ἢ $\Gamma=90^0$.

Λύσις. Ἡ (4) γράφεται (ἄσκησις 91,1), ὡς ἑξῆς:

$$-4\eta\mu 2A\eta\mu 2B\eta\mu 2\Gamma=0.$$

Ἄρα ἢ $\eta\mu 2A=0 \implies 2A=k\cdot 180^0 \implies A=k\cdot 90^0$.

καὶ ἐπειδὴ $0<2A<360^0 \implies 0^0<k\cdot 180^0<360^0$ ἢ $0<k<2 \implies k=1$.

Ἄρα $A=k\cdot 90^0=1\cdot 90^0=90^0$.

Ὁμοίως ἢ $B=90^0$ ἢ $\Gamma=90^0$.

103. Ἐὰν $\eta\mu 3A+\eta\mu 3B+\eta\mu 3\Gamma=0$, τότε ἡ μία τῶν γωνιῶν τοῦ τριγώνου ΑΒΓ εἶναι 60^0.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu 3\Gamma=\eta\mu[540^0-3(A+B)]=\eta\mu 3(A+B)=2\eta\mu\frac{3}{2}(A+B)\sigma\upsilon\nu\frac{3}{2}(A+B)]$$

καί:

$$\eta\mu 3A+\eta\mu 3B+\eta\mu 3\Gamma=2\eta\mu\frac{3}{2}(A+B)\sigma\upsilon\nu\frac{3}{2}(A-B)+$$

$$+2\eta\mu\frac{3}{2}(A+B)\sigma\upsilon\nu\frac{3}{2}(A+B)=$$

$$=2\eta\mu\frac{3}{2}(A+B)[\sigma\upsilon\nu\frac{3}{2}(A-B)+\sigma\upsilon\nu\frac{3}{2}(A+B)]$$

$$=4\sigma\upsilon\nu\frac{3}{2}\Gamma\ \sigma\upsilon\nu\frac{3}{2}A\ \sigma\upsilon\nu\frac{3}{2}B=0.$$

Ἔστω ὅτι $\sigma\upsilon\nu\frac{3}{2}A=0 \implies \frac{3}{2}A=90^0$.

Ἐπειδὴ $0^0<\frac{3}{2}A<270^0$, ἕπεται ὅτι $3A=180^0$ ἢ $A=60^0$.

104. Ἐὰν $\eta\mu\frac{A}{2}\sigma\upsilon\nu^3\frac{B}{2}=\eta\mu\frac{B}{2}\sigma\upsilon\nu^3\frac{A}{2}$, τοῦτο τὸ τρίγωνον ΑΒΓ εἶναι ἰσοσκελές·

Λύσις. Διαιροῦμεν ἀμφότερα τὰ μέλη τῆς δοθείσης σχέσεως διὰ $\sigma\upsilon\nu\frac{A}{2}\sigma\upsilon\nu\frac{B}{2}$ καὶ λαμβάνομεν:

$$\varepsilon\varphi\frac{A}{2}\sigma\upsilon\nu^2\frac{B}{2}=\varepsilon\varphi\frac{B}{2}\sigma\upsilon\nu^2\frac{A}{2}$$

ἢ

$$\varepsilon\varphi\frac{A}{2}\cdot\frac{1}{1+\varepsilon\varphi^2\frac{B}{2}}=\varepsilon\varphi\frac{B}{2}\cdot\frac{1}{1+\varepsilon\varphi^2\frac{A}{2}}$$

ἤ $$(\varepsilon\varphi^3\frac{A}{2}-\varepsilon\varphi^3\frac{B}{2}+\varepsilon\varphi\frac{A}{2}-\varepsilon\varphi\frac{B}{2}=0.$$

ἤ $$\left(\varepsilon\varphi\frac{A}{2}-\varepsilon\varphi\frac{B}{2}\right)\left(\varepsilon\varphi^2\frac{A}{2}+\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2}+\varepsilon\varphi^2\frac{B}{2}+1\right)=0.$$

Ἐπειδή, προφανῶς, ὁ δεύτερος παράγων εἶναι θετικός, ἔπεται:

$$\varepsilon\varphi\frac{A}{2}-\varepsilon\varphi\frac{B}{2}=0 \implies \frac{A}{2}=\frac{B}{2} \implies A=B,$$

καθόσον $$\frac{A}{2}<90^0 \text{ καὶ } \frac{B}{2}<90^0.$$

105. Ἐὰν συν3Α+συν3Β+συν3Γ=1, τότε ἡ μία γωνία τοῦ τριγώνου ΑΒΓ εἶναι 120°.

Λύσις. Ἐπειδὴ $3\Gamma=3\pi-3(A+B) \implies \sigma\upsilon\nu 3\Gamma=-\sigma\upsilon\nu 3(A+B)$ καὶ ἡ δοθεῖσα σχέσις γράφεται:

$$2\sigma\upsilon\nu\frac{3}{2}(A+B)\sigma\upsilon\nu\frac{3}{2}(A-B)-1-\sigma\upsilon\nu 3(A+B)-0$$

ἤ $$2\sigma\upsilon\nu\frac{3}{2}(A+B)\sigma\upsilon\nu\frac{3}{2}(A-B)-2\sigma\upsilon\nu^2\frac{3}{2}(A+B)=0$$

ἤ $$\sigma\upsilon\nu\frac{3}{2}(A+B)\left[\sigma\upsilon\nu\frac{3}{2}(A-B)-\sigma\upsilon\nu\frac{3}{2}(A+B)\right]=0$$

ἤ $$\eta\mu\frac{3A}{2}\,\eta\mu\frac{3B}{2}\,\eta\mu\frac{3\Gamma}{2}=0.$$

Ὅθεν ἤ $$\eta\mu\frac{3A}{2}=0 \implies \frac{3A}{2}=k\pi \implies A=2k\cdot\frac{\pi}{3}.$$

Ἐπειδὴ $$0<A<\pi \implies A=\frac{2\pi}{3}=120^0$$

ἤ $$\eta\mu\frac{3B}{2}=0 \implies B=120^0 \text{ ἢ } \eta\mu\frac{3\Gamma}{2}=0 \implies \Gamma=120^0.$$

106. Ἐὰν x+y+ω=xyω, νὰ ἀποδειχθῇ ὅτι:

1. $$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2\omega}{1-\omega^2}=\frac{2y}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2\omega}{1-\omega^2}.$$

Λύσις. Θέτομεν $x=\varepsilon\varphi A$, $y=\varepsilon\varphi B$, $\omega=\varepsilon\varphi\Gamma$, ὁπότε

$$\varepsilon\varphi A+\varepsilon\varphi B+\varepsilon\varphi\Gamma=\varepsilon\varphi A\varepsilon\varphi B\varepsilon\varphi\Gamma \text{ ἢ } \frac{\varepsilon\varphi A+\varepsilon\varphi B}{1-\varepsilon\varphi A\varepsilon\varphi B}=-\varepsilon\varphi\Gamma \text{ ἢ } \varepsilon\varphi(A+B)=\varepsilon\varphi(\pi-\Gamma)$$

ἐξ οὗ $$A+B+\Gamma=\nu\pi+\pi,\quad \nu\in\mathbf{N}.$$

Ἄρα:

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2\omega}{1-\omega^2}=\frac{2\varepsilon\varphi A}{1-\varepsilon\varphi^2 A}+\frac{2\varepsilon\varphi B}{1-\varepsilon\varphi^2 B}+\frac{2\varepsilon\varphi\Gamma}{1-\varepsilon\varphi^2\Gamma}=$$

$$=\varepsilon\varphi 2A+\varepsilon\varphi 2B+\varepsilon\varphi\Gamma=\varepsilon\varphi 2A\varepsilon\varphi 2B\varepsilon\varphi 2\Gamma=\frac{2x}{1-x^2}\cdot\frac{2y}{1-y^2}\cdot\frac{2\omega}{1-\omega^2}.$$

2. $\frac{3x-x^3}{1-3x^2}+\frac{3y-x^3}{1-3y^2}+\frac{3\omega-\omega^3}{1-3\omega^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3\omega-\omega^3}{1-3\omega^2}$.

Λύσις. Ἐὰν θέσωμεν $x=\varepsilon\varphi A$, $y=\varepsilon\varphi B$, $\omega=\varepsilon\varphi\Gamma$, τότε:

$$A+B+\Gamma=\nu\pi+\pi,\quad \nu\in\mathbf{N}$$

καὶ

$$\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3\omega-\omega^2}{1-3\omega^2}=$$

$$\frac{3\varepsilon\varphi A-\varepsilon\varphi^3 A}{1-3\varepsilon\varphi^2 A}+\frac{3\varepsilon\varphi B-\varepsilon\varphi^3 B}{1-3\varepsilon\varphi^2 B}+\frac{3\varepsilon\varphi\Gamma-\varepsilon\varphi^3\Gamma}{1-3\varepsilon\varphi^2\Gamma}=$$

$$\varepsilon\varphi 3A+\varepsilon\varphi 3B+\varepsilon\varphi 3\Gamma=\varepsilon\varphi 3A\varepsilon\varphi 3B\varepsilon\varphi 3\Gamma=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3\omega-\omega^3}{1-3\omega^2}.$$

3. $\Sigma\, x(1-y^2)(1-\omega^2)=4xy\omega.$

Λύσις. Ἐὰν θέσωμεν $x=\varepsilon\varphi\alpha$, $y=\varepsilon\varphi\beta$, $\omega=\varepsilon\varphi\gamma$, τότε

$$\alpha+\beta+\gamma=\nu\pi+\pi,\quad \nu\in\mathbf{N}$$

καὶ $\quad 2\alpha+2\beta+2\gamma=2\nu\pi+2\pi$, ὁπότε

$$\varepsilon\varphi 2\alpha+\varepsilon\varphi 2\beta+\varepsilon\varphi 2\gamma=\varepsilon\varphi 2\alpha\varepsilon\varphi 2\beta\varepsilon\varphi 2\gamma$$

ἢ

$$\frac{2x}{1-x^2}+\frac{2y}{1-y^2}+\frac{2\omega}{1-\omega^2}=\frac{8xy\omega}{(1-x^2)(1-y^2)(1-\omega^2)}$$

ἐξ οὗ:

$$\Sigma\, x(1-y^2)(1-\omega^2)=4xy\omega.$$

107. Ἐὰν $\alpha=\beta+\gamma$, νὰ ἀποδειχθῇ ὅτι:
$\eta\mu(\alpha+\beta+\gamma)+\eta\mu(\alpha+\beta-\gamma)+\eta\mu(\alpha-\beta+\gamma)=4\eta\mu\alpha\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\eta\mu(\alpha+\beta+\gamma)+\eta\mu(\alpha+\beta-\gamma)+\eta\mu(\alpha-\beta+\gamma)=$$
$$=\eta\mu 2\alpha+2\eta\mu\alpha\sigma\upsilon\nu(\beta-\gamma)=2\eta\mu\alpha\sigma\upsilon\nu\alpha+2\eta\mu\alpha\sigma\upsilon\nu(\beta-\gamma)=$$
$$=2\eta\mu\alpha\sigma\upsilon\nu(\beta+\gamma)+2\eta\mu\alpha\sigma\upsilon\nu(\beta-\gamma)=$$
$$=2\eta\mu\alpha[\sigma\upsilon\nu(\beta+\gamma)+\sigma\upsilon\nu(\beta-\gamma)]=4\eta\mu\alpha\sigma\upsilon\nu\beta\sigma\upsilon\nu\gamma.$$

108. Ἐὰν $\alpha+\beta+\gamma=0$, νὰ ἀποδειχθῇ ὅτι:
$\eta\mu 2\alpha+\eta\mu 2\beta+\eta\mu 2\gamma=2(\eta\mu\alpha+\eta\mu\beta+\eta\mu\gamma)(1+\sigma\upsilon\nu\alpha+\sigma\upsilon\nu\beta+\sigma\upsilon\nu\gamma)$.

Λύσις. Ἔχομεν $\alpha+\beta+\gamma=0 \Longrightarrow \alpha+\beta=-\gamma$ ἢ $\eta\mu(\alpha+\beta)=-\eta\mu\gamma$ καὶ $\sigma\upsilon\nu(\alpha+\beta)=\sigma\upsilon\nu\gamma$. Ὅθεν:

$$\eta\mu 2\alpha+\eta\mu 2\beta+\eta\mu 2\gamma=2\eta\mu(\alpha+\beta)\sigma\upsilon\nu(\alpha-\beta)+2\eta\mu\gamma\sigma\upsilon\nu\gamma=$$
$$=-2\eta\mu\gamma\sigma\upsilon\nu(\alpha-\beta)+2\eta\mu\gamma\sigma\upsilon\nu(\alpha+\beta)$$
$$=-2\eta\mu\gamma[\sigma\upsilon\nu(\alpha-\beta)-\sigma\upsilon\nu(\alpha+\beta)]=-4\eta\mu\alpha\eta\mu\beta\eta\mu\gamma=$$
$$=-32\eta\mu\frac{\alpha}{2}\,\eta\mu\frac{\beta}{2}\,\eta\mu\frac{\gamma}{2}\,\sigma\upsilon\nu\frac{\alpha}{2}\,\sigma\upsilon\nu\frac{\beta}{2}\,\sigma\upsilon\nu\frac{\gamma}{2}$$

$$=-8\left[\left(\sigma\upsilon\nu\frac{\alpha-\beta}{2}-\sigma\upsilon\nu\frac{\alpha+\beta}{2}\right)\eta\mu\frac{\gamma}{2}\right]\left[\left(\sigma\upsilon\nu\frac{\alpha+\beta}{2}+\sigma\upsilon\nu\frac{\alpha-\beta}{2}\right)\sigma\upsilon\nu\frac{\gamma}{2}\right]$$

$$=8\left[\sigma\upsilon\nu\frac{\alpha-\beta}{2}\,\eta\mu\frac{\alpha+\beta}{2}+\eta\mu\frac{\gamma}{2}\,\sigma\upsilon\nu\frac{\gamma}{2}\right]\left[\sigma\upsilon\nu^2\frac{\gamma}{2}+\sigma\upsilon\nu\frac{\alpha+\beta}{2}\,\sigma\upsilon\nu\frac{\alpha-\beta}{2}\right]$$

$$=2(\eta\mu\alpha+\eta\mu\beta+\eta\mu\gamma)(1+\sigma\upsilon\nu\gamma+\sigma\upsilon\nu\alpha+\sigma\upsilon\nu\beta).$$

109. Ἐὰν $\eta\mu\alpha+\eta\mu\beta=\eta\mu(\alpha+\beta)$, νὰ ἀποδειχθῇ ὅτι :
$\alpha=2k\pi,\quad \beta=2k_1\pi,\quad \alpha+\beta=2k_2\pi\quad (k, k_1, k_2\in N)$.

Λύσις. Ἔχομεν : $\eta\mu\alpha+\eta\mu\beta=\eta\mu(\alpha+\beta)$ ἢ

$$2\eta\mu\frac{\alpha+\beta}{2}\,\sigma\upsilon\nu\frac{\alpha-\beta}{2}=2\eta\mu\frac{\alpha+\beta}{2}\,\sigma\upsilon\nu\frac{\alpha+\beta}{2}$$

ἢ $$\eta\mu\frac{\alpha+\beta}{2}\left[\sigma\upsilon\nu\frac{\alpha-\beta}{2}-\sigma\upsilon\nu\frac{\alpha+\beta}{2}\right]=0$$

ἢ $$2\eta\mu\frac{\alpha+\beta}{2}\,\eta\mu\frac{\alpha}{2}\,\eta\mu\frac{\beta}{2}=0,\quad \text{ἐξ οὗ}$$

ἢ $$\eta\mu\frac{\alpha}{2}=0\implies\frac{\alpha}{2}=k\pi\implies\alpha=2k\pi$$

ἢ $$\eta\mu\frac{\beta}{2}=0\implies\frac{\beta}{2}=k_1\pi\implies\beta=2k_1\pi$$

ἢ $$\eta\mu\frac{\alpha+\beta}{2}=0\implies\frac{\alpha+\beta}{2}=k_2\pi\implies\alpha+\beta=2k_2\pi.$$

110. Ἐὰν $\eta\mu\alpha+\eta\mu\beta+\eta\mu\gamma=\eta\mu(\alpha+\beta+\gamma)$, τότε :
$\alpha+\beta=2k\pi,\quad \beta+\gamma=2k_1\pi,\quad \gamma+\alpha=2k_2\pi\quad (k, k_1, k_2\in N)$

Λύσις. Ἔχομεν : $\eta\mu\alpha+\eta\mu\beta+\eta\mu\gamma-\eta\mu(\alpha+\beta+\gamma)=0$ ἢ

$$4\eta\mu\frac{\alpha+\beta}{2}\,\eta\mu\frac{\beta+\gamma}{2}\,\eta\mu\frac{\gamma+\alpha}{2}=0$$

ὁπότε ἢ $$\eta\mu\frac{\alpha+\beta}{2}=0\implies\frac{\alpha+\beta}{2}=k\pi\implies\alpha+\beta=2k\pi,$$

ἢ $$\eta\mu\frac{\beta+\gamma}{2}=0\implies\frac{\beta+\gamma}{2}=k_1\pi\implies\beta+\gamma=2k_1\pi,$$

ἢ $$\eta\mu\frac{\gamma+\alpha}{2}=0\implies\frac{\gamma+\alpha}{2}=k_2\pi\implies\gamma+\alpha=2k_2\pi.$$

111. Ἐὰν $\eta\mu(\alpha-\beta)=\eta\mu^2\alpha-\eta\mu^2\beta$, τότε :
ἢ $\alpha-\beta=k\pi$ ἢ $\alpha+\beta=2k_1\pi+\dfrac{\pi}{2}$. $(k, k_1\in N)$

Λύσις. Ἔχομεν : $\eta\mu(\alpha-\beta)=\eta\mu^2\alpha-\eta\mu^2\beta=\eta\mu(\beta+\gamma)\eta\mu(\beta-\gamma)$

ἢ $\eta\mu(\alpha-\beta)[1-\eta\mu(\beta-\gamma)]=0,$ ἐξ οὗ ἢ $\eta\mu(\alpha-\beta)=0 \Longrightarrow \alpha-\beta=k\pi$

ἢ $1-\eta\mu(\beta-\gamma)=0 \Longrightarrow \eta\mu(\beta-\gamma)=1=\eta\mu\frac{\pi}{2} \Longrightarrow \beta-\gamma=2k_1\pi+\frac{\pi}{2}.$

112. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

$$1+\Sigma\frac{\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu A\cdot\eta\mu^2 B}=(\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma)^2.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu A\eta\mu^2 B}=\frac{\eta\mu\Gamma}{\eta\mu A\eta\mu B}\cdot\sigma\varphi B=\frac{\eta\mu(A+B)}{\eta\mu A\eta\mu B}\cdot\sigma\varphi B=$$

$$=\frac{\eta\mu A\sigma\upsilon\nu B+\eta\mu B\sigma\upsilon\nu A}{\eta\mu A\eta\mu B}\cdot\sigma\varphi B=(\sigma\varphi A+\sigma\varphi B)\sigma\varphi B=\sigma\varphi A\cdot\sigma\varphi B+\sigma\varphi^2 B.$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν γραμμάτων Α, Β, Γ, ἔχομεν :

$$1+\Sigma\frac{\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu A\eta\mu^2 B}=1+(\sigma\varphi A\sigma\varphi B+\sigma\varphi^2 B)+(\sigma\varphi B\sigma\varphi\Gamma+\sigma\varphi^2\Gamma)+(\sigma\varphi\Gamma\sigma\varphi A+\sigma\varphi^2 A)=$$

$$=\sigma\varphi A\sigma\varphi B+\sigma\varphi B\sigma\varphi\Gamma+\sigma\varphi\Gamma\sigma\varphi A+\sigma\varphi A\sigma\varphi B+\sigma\varphi B\sigma\varphi\Gamma+\sigma\varphi A\sigma\varphi\Gamma+$$
$$+\sigma\varphi^2 A+\sigma\varphi^2 B+\sigma\varphi^2\Gamma=$$
$$=\sigma\varphi^2 A+\sigma\varphi^2 B+\sigma\varphi^2\Gamma+2\sigma\varphi A\sigma\varphi B+2\sigma\varphi B\sigma\varphi\Gamma+2\sigma\varphi\Gamma\sigma\varphi A=$$
$$=(\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma)^2.$$

113. Ἐὰν $\nu\in N$ καὶ $A+B+\Gamma=180^\circ$, νὰ ἀποδειχθῇ ὅτι :
$\eta\mu(2\nu A)+\eta\mu(2\nu B)+\eta\mu(2\nu\Gamma)=4(-1)^{\nu-1}\eta\mu(\nu A)\eta\mu(\nu B)\eta\mu(\nu\Gamma)$.

Λύσις. Ἔχομεν διαδοχικῶς :

$$\eta\mu(2\nu A)+\eta\mu(2\nu B)+\eta\mu(2\nu\Gamma)=2\eta\mu\cdot(\nu A+\nu B)\sigma\upsilon\nu(\nu A-\nu B)+2\eta\mu(\nu\Gamma)\sigma\upsilon\nu(\nu\Gamma)\quad(1)$$

Ἐπειδὴ $A+B+\Gamma=\pi \Longrightarrow \nu A+\nu B=\nu\pi-\nu\Gamma$, ὁπότε

$$\eta\mu(\nu A+\nu B)=\eta\mu(\nu\pi-\nu\Gamma)=-\eta\mu(\nu\Gamma),\quad \text{ἂν } \nu=\text{ἄρτιος}$$

καὶ $$\eta\mu(\nu A+\nu B)=\eta\mu(\nu\pi-\nu\Gamma)=\eta\mu(\nu\Gamma),\quad \text{ἂν } \nu=\text{περιττός},$$

Ἄρα, διὰ $\nu=$ἄρτιος, ἡ (1) γίνεται :

$$\Sigma\,\eta\mu(2\nu A)=-2\eta\mu(\nu\Gamma)\sigma\upsilon\nu(\nu A-\nu B)+2\eta\mu(\nu\Gamma)\sigma\upsilon\nu(\nu\Gamma)$$
$$=-2\eta\mu(\nu\Gamma)[\sigma\upsilon\nu(\nu A-\nu B)-\sigma\upsilon\nu(\nu\Gamma)]$$
$$=-2\eta\mu(\nu\Gamma)[\sigma\upsilon\nu(\nu A-\nu B)-\sigma\upsilon\nu(\nu A+\nu B)]$$
$$=-4\eta\mu(\nu A)\eta\mu(\nu B)\eta\mu(\nu\Gamma).$$

Ἐὰν δὲ $\nu=$περιττός, τότε :

$$\Sigma\,\eta\mu(2\nu A)=4\eta\mu(\nu A)\eta\mu(\nu B)\eta\mu(\nu\Gamma)$$

ὁπότε : $$\Sigma\,\eta\mu(2\nu A)=4(-1)^{\nu-1}\eta\mu(\nu A)\eta\mu(\nu B)\eta\mu(\nu\Gamma).$$

ΚΕΦΑΛΑΙΟΝ IV

ΕΦΑΡΜΟΓΑΙ ΤΩΝ ΤΡΙΓΩΝΟΜΕΤΡΙΚΩΝ ΜΕΤΑΣΧΗΜΑΤΙΣΜΩΝ ΣΧΕΣΕΙΣ ΜΕΤΑΞΥ ΤΩΝ ΚΥΡΙΩΝ ΣΤΟΙΧΕΙΩΝ ΤΡΙΓΩΝΟΥ

114. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $\Gamma = 120^\circ$ καὶ $2\alpha = \beta(\sqrt{3} - 1)$, νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου τούτου.

Λύσις. Ἐπειδὴ $2\alpha = \beta(\sqrt{3} - 1)$, ἕπεται

$$2\cdot 2R\eta\mu A = (2R\eta\mu B)(\sqrt{3}-1) \quad \text{ἢ} \quad 2\eta\mu A = (\eta\mu B)(\sqrt{3}-1)$$

$$\text{ἢ} \quad \frac{\eta\mu A}{\eta\mu B} = \frac{\sqrt{3}-1}{2} \quad \text{ἢ} \quad \frac{\eta\mu A + \eta\mu B}{\eta\mu A - \eta\mu B} = \frac{\sqrt{3}-1+2}{\sqrt{3}-1-2} = \frac{\sqrt{3}+1}{\sqrt{3}-3}$$

$$\text{ἢ} \quad \frac{2\eta\mu \frac{A+B}{2} \sigma\upsilon\nu \frac{A-B}{2}}{2\eta\mu \frac{A-B}{2} \sigma\upsilon\nu \frac{A+B}{2}} = \frac{\sqrt{3}+1}{\sqrt{3}-3} = \frac{(\sqrt{3}+1)(\sqrt{3}+3)}{3-9} = \frac{6+4\sqrt{3}}{-6} = \frac{3+2\sqrt{3}}{-3}$$

$$\varepsilon\varphi \frac{A+B}{2} \sigma\varphi \frac{A-B}{2} = \frac{3+2\sqrt{3}}{-3} \quad \text{ἢ} \quad \sigma\varphi \frac{\Gamma}{2} \sigma\varphi \frac{A-B}{2} = \frac{3+2\sqrt{3}}{-3}$$

$$\text{ἢ} \quad \frac{\sqrt{3}}{3} \sigma\varphi \frac{A-B}{2} = \frac{3+2\sqrt{3}}{-3} \Longrightarrow \sigma\varphi \frac{A-B}{2} =$$

$$\frac{3(3+2\sqrt{3})}{-3\sqrt{3}} = \frac{3+2\sqrt{3}}{-\sqrt{3}} = -(2+\sqrt{3}) = \sigma\varphi(-15^\circ)$$

ἐξ οὗ: $\frac{A-B}{2} = -15^\circ$ ἢ $A - B = -30^\circ$. Ἐπειδὴ δὲ $A + B = 60^\circ$, ἕπεται ὅτι: $2A = 30^\circ \Longrightarrow A = 15^\circ$, ὅτε $B = 45^\circ$.

115. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $3\alpha = (\beta+\gamma)\sqrt{3}$ καὶ $A = 60^\circ$, νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου τούτου.

Λύσις. Ἐκ τῶν τύπων τοῦ Mollweide γνωρίζομεν ὅτι:

$$\frac{\beta+\gamma}{\alpha} \eta\mu \frac{A}{2} = \sigma\upsilon\nu \frac{B-\Gamma}{2}. \quad \text{Ἀλλὰ} \quad \frac{\beta+\gamma}{\alpha} = \frac{3}{\sqrt{3}}. \quad \text{Ἄρα:}$$

$$\sigma\upsilon\nu \frac{B-\Gamma}{2} = \frac{3}{\sqrt{3}} \eta\mu \frac{A}{2} = \sqrt{3}\cdot\eta\mu 30^\circ = \sqrt{3}\,\frac{1}{2} = \frac{\sqrt{3}}{2} = \sigma\upsilon\nu 30^\circ$$

ἐξ οὗ: $\frac{B-\Gamma}{2}=30^\circ \Rightarrow B-\Gamma=60^\circ$. Ἀλλὰ $B+\Gamma=180^\circ-60^\circ=120^\circ$.

Ἄρα $2B=180^\circ \Rightarrow B=90^\circ$, ὅτε $\Gamma=30^\circ$.

116. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι β=2γ καὶ Α=60°, νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου τούτου.

Λύσις. Ἡ δοθεῖσα σχέσις $\beta=2\gamma$ γράφεται:

$$\frac{\beta}{\gamma}=\frac{2}{1} \quad ἢ \quad \frac{\beta-\gamma}{\beta+\gamma}=\frac{2-1}{2+1}=\frac{1}{3}.$$

Ἐκ τοῦ τύπου $\frac{\beta-\gamma}{\beta+\gamma}\,\sigma\varphi\,\frac{A}{2}=\epsilon\varphi\,\frac{B-\Gamma}{2}$ τοῦ Mollweide, ἔχομεν:

$$\epsilon\varphi\frac{B-\Gamma}{2}=\frac{1}{3}\,\sigma\varphi\,\frac{A}{2}=\frac{1}{3}\cdot\sigma\varphi 30^\circ=\frac{1}{3}\cdot\sqrt{3}=\frac{\sqrt{3}}{3}=\sigma\varphi 60^\circ=\epsilon\varphi 30^\circ.$$

Ἄρα $\frac{B-\Gamma}{2}=30^\circ \Rightarrow B-\Gamma=60^\circ$. Ἀλλὰ $B+\Gamma=120^\circ$.

Ἄρα $2B=180^\circ \Rightarrow B=90^\circ$. ὁπότε $\Gamma=30^\circ$.

117. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι β=α(√3 −1) καὶ Γ=30°, νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου τούτου.

Λύσις. Ἐκ τῆς $\beta=\alpha(\sqrt{3}-1)$ λαμβάνομεν:

$$\frac{\alpha}{\beta}=\frac{1}{\sqrt{3}-1} \Rightarrow \frac{\alpha-\beta}{\alpha+\beta}=\frac{1-\sqrt{3}+1}{1+\sqrt{3}-1}=\frac{2-\sqrt{3}}{\sqrt{3}}$$ καὶ βάσει τοῦ τύπου (83)

$$\epsilon\varphi\,\frac{A-B}{2}=\frac{\alpha-\beta}{\alpha+\beta}\,\sigma\varphi\,\frac{\Gamma}{2}=\frac{2-\sqrt{3}}{\sqrt{3}}\cdot\sigma\varphi 15^\circ=$$

$$=\frac{2-\sqrt{3}}{\sqrt{3}}\cdot(2+\sqrt{3})=\frac{4-3}{\sqrt{3}}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}=\epsilon\varphi 30^\circ$$

ἐξ οὗ: $\frac{A-B}{2}=30^\circ \Rightarrow A-B=60^\circ$. Ἀλλὰ $A+B=150^\circ$. ὁπότε:

$$2A=210^\circ \Rightarrow A=105^\circ, \quad καὶ \quad B=45^\circ.$$

Σημείωσις: Γνωρίζομεν ὅτι:

$$\gamma^2=\alpha^2+\beta^2-2\alpha\beta\sigma\upsilon\nu\Gamma=\alpha^2+(4-2\sqrt{3})\alpha^2-\alpha^2(\sqrt{3}-1)=(2-\sqrt{3})\alpha^2$$

ἢ $\quad \alpha^2=(2+\sqrt{3})\gamma^2 \quad$ ἢ $\quad \eta\mu^2 A=(2+\sqrt{3})\eta\mu^2\Gamma=(2+\sqrt{3})\cdot\frac{1}{4}$

ἢ $\quad \eta\mu^2 A=\frac{2+\sqrt{3}}{4}=\frac{4+2\sqrt{3}}{8} \Rightarrow \eta\mu A=\frac{\sqrt{3}+1}{2\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4}=\eta\mu 75^\circ$

ἢ $\eta\mu A = \dfrac{\sqrt{6}+\sqrt{2}}{4} = \eta\mu 105^0$. Ἄρα $A=75^0$, ὁπότε $B=75^0$

ἢ $A=105^0$ ὁπότε $B=45^0$. Ἀλλὰ $A=75^0=B$ ἀποκλείεται διότι τότε θὰ ἦτο $\alpha=\beta$, ὅπερ ἄτοπον, διότι $\beta=\alpha(\sqrt{3}-1)$·

118. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $\alpha=2$, $\gamma=\sqrt{2}$, $B=15^0$ νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\beta^2=\alpha^2+\gamma^2-2\alpha\gamma\cdot\sigma\upsilon\nu B=4+2-2\cdot2\cdot\sqrt{2}\cdot\frac{\sqrt{6}+\sqrt{2}}{4}=4-2\sqrt{3}=(\sqrt{3}-1)^2$$

ὅθεν $\beta=\sqrt{3}-1$ καὶ κατ' ἀκολουθίαν:

$$\eta\mu A=\frac{\alpha}{\beta}\,\eta\mu B=\frac{2}{\sqrt{3}-1}\cdot\frac{\sqrt{6}-\sqrt{2}}{4}=\frac{\sqrt{2}}{2}=\eta\mu 45^0=\eta\mu 135^0.$$

Ἐπειδὴ $\alpha>\gamma>\beta$, ἔπεται $A=135^0$, ὁπότε $\Gamma=30^0$.

119. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $A=45^0$ καὶ $\dfrac{\beta}{\gamma}=\dfrac{\sqrt{2}}{\sqrt{3}+1}$, νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου τούτου.

Λύσις. Ἔχομεν:

$$\frac{\beta}{\gamma}=\frac{\sqrt{2}}{\sqrt{3}+1}=\frac{\sqrt{2}(\sqrt{3}-1)}{3-1}=\frac{\sqrt{6}-\sqrt{2}}{2}\Longrightarrow\beta=\frac{(\sqrt{6}-\sqrt{2})\gamma}{2}$$

καὶ $$\alpha^2=\beta^2+\gamma^2-2\beta\gamma\sigma\upsilon\nu A=\left(\frac{\sqrt{6}-\sqrt{2}}{2}\right)^2\gamma^2+\gamma^2-2\cdot\frac{\sqrt{6}-\sqrt{2}}{2}\gamma^2\cdot\frac{\sqrt{2}}{2}$$

$$=\frac{6+2-4\sqrt{3}}{4}\gamma^2+\gamma^2-\frac{2\sqrt{3}-2}{2}\gamma^2=(2-\sqrt{3})\gamma^2+\gamma^2-(\sqrt{3}-1)\gamma^2=$$

$$=\gamma^2(2-\sqrt{3}+1-\sqrt{3}+1)=(4-2\sqrt{3})\gamma^2\Longrightarrow$$

$$\gamma^2=\frac{\alpha^2}{4-2\sqrt{3}}=\frac{\alpha^2(4+2\sqrt{3})}{16-12}=\frac{\alpha^2(2+\sqrt{3})}{2}$$

ἢ $$\eta\mu^2\Gamma=\frac{4+2\sqrt{3}}{4}\,\eta\mu^2 A=\frac{4+2\sqrt{3}}{4}\cdot\frac{2}{4}=\frac{8+4\sqrt{3}}{4\cdot4}=\frac{2+\sqrt{3}}{4}$$

ἐξ οὗ $$\eta\mu\Gamma=\frac{\sqrt{2+\sqrt{3}}}{2}=\frac{\sqrt{6}+\sqrt{2}}{4}=\eta\mu 75^0=\eta\mu 105^0.$$

Ὅθεν ἢ $\Gamma=75^0$ ἢ $\Gamma=105^0$, ὁπότε $B=60^0$ ἢ $B=30^0$.

120. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $B=135^0$ καὶ $\dfrac{\alpha}{\beta}=\dfrac{\sqrt{6}}{2}$, νὰ ὑπολογισθοῦν αἱ ἄλλαι γωνίαι τοῦ τριγώνου τούτου.

Λύσις. Ἔχομεν $\alpha=\dfrac{\beta\sqrt{6}}{2}$ καὶ $\eta\mu A=\dfrac{\sqrt{6}}{2}\cdot\eta\mu B$

ἢ $\quad ημA=\frac{\sqrt{6}}{2}\cdot ημ135^0=\frac{\sqrt{6}}{2}\cdot\frac{\sqrt{2}}{2}=\frac{\sqrt{3}}{2}=ημ60^0=ημ120^0.$

Ὅθεν $\quad A=60^0$ καὶ $A=120^0$. Ἐπειδὴ δὲ $B=135^0$, ἕπεται

ὅτι $\quad A+B=135^0+60^0=195^0$ ἀδύνατον

καὶ $\quad A+B=120^0+135^0=255^0$ ἀδύνατον.

Ὥστε τὸ πρόβλημα εἶναι ἀδύνατον

121. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $$\mathbf{α(βσυνΓ-γσυνB)=β^2-γ^2.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$α(βσυνΓ-γσυνB)=2RημA(2RημBσυνΓ-2RημΓσυνB)=$$
$$=4R^2ημ(B+Γ)ημ(B-Γ)$$
$$=4R^2(ημ^2B-ημ^2Γ)=4R^2ημ^2B-4R^2ημ^2Γ=β^2-γ^2.$$

2. $$\mathbf{α(συνB+συνΓ)=2(β+γ)ημ^2\frac{A}{2}.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$α(συνB+συνΓ)=2RημA\cdot 2συν\frac{B+Γ}{2}συν\frac{B-Γ}{2}$$
$$=2R\cdot 2ημ\frac{A}{2}συν\frac{A}{2}\cdot 2ημ\frac{A}{2}\cdot συν\frac{B-Γ}{2}$$
$$=8Rημ^2\frac{A}{2}\cdot συν\frac{A}{2}συν\frac{B-Γ}{2}=8Rημ^2\frac{A}{2}\cdot ημ\frac{B+Γ}{2}συν\frac{B-Γ}{2}$$
$$=4R\cdot 2ημ\frac{B+Γ}{2}συν\frac{B-Γ}{2}\cdot ημ^2\frac{A}{2}$$
$$=4R(ημB+ημΓ)ημ^2\frac{A}{2}$$
$$=2(2RημB+2RημΓ)ημ^2\frac{A}{2}=2(β+γ)ημ^2\frac{A}{2}.$$

3. $$\mathbf{α(συνΓ-συνB)=2(β-γ)συν^2\frac{A}{2}.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$α(συνΓ-συνB)=2RημA2ημ\frac{Γ+B}{2}ημ\frac{B-Γ}{2}=$$
$$=2R\cdot 2ημ\frac{A}{2}\cdot συν\frac{A}{2}2συν\frac{A}{2}ημ\frac{B-Γ}{2}=$$
$$=4R\cdot 2ημ\frac{A}{2}ημ\frac{B-Γ}{2}\cdot συν^2\frac{A}{2}=4R\cdot 2συν\frac{B+Γ}{2}ημ\frac{B-Γ}{2}συν^2\frac{A}{2}$$
$$=4R\cdot(ημB-ημΓ)συν^2\frac{A}{2}=2(2RημB-2RημΓ)συν^2\frac{A}{2}=$$
$$=2(β-γ)συν^2\frac{A}{2}.$$

4. $$\alpha\eta\mu\left(\frac{A}{2}+B\right)=(\beta+\gamma)\eta\mu\frac{A}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\alpha\eta\mu\left(\frac{A}{2}+B\right)=\alpha\eta\mu\left[90^0+\frac{B-\Gamma}{2}\right]=\alpha\sigma\upsilon\nu\frac{B-\Gamma}{2}=2R\eta\mu A\sigma\upsilon\nu\frac{B-\Gamma}{2}=$$

$$=2R\cdot 2\eta\mu\frac{A}{2}\sigma\upsilon\nu\frac{A}{2}\sigma\upsilon\nu\frac{B-\Gamma}{2}=2R\cdot 2\eta\mu\frac{A}{2}\cdot\eta\mu\frac{B+\Gamma}{2}\sigma\upsilon\nu\frac{B-\Gamma}{2}$$

$$=2R\cdot 2\eta\mu\frac{B+\Gamma}{2}\sigma\upsilon\nu\frac{B-\Gamma}{2}\cdot\eta\mu\frac{A}{2}=2R(\eta\mu B+\eta\mu\Gamma)\eta\mu\frac{A}{2}=$$

$$=(2R\eta\mu B+2R\eta\mu\Gamma)\eta\mu\frac{A}{2}=(\beta+\gamma)\eta\mu\frac{A}{2}.$$

5. $$\beta\sigma\upsilon\nu B+\gamma\sigma\upsilon\nu\Gamma=\alpha\sigma\upsilon\nu(B-\Gamma).$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\beta\sigma\upsilon\nu B+\gamma\sigma\upsilon\nu\Gamma=2R\eta\mu B\sigma\upsilon\nu B+2R\eta\mu\Gamma\sigma\upsilon\nu\Gamma=$$

$$=R(2\eta\mu B\sigma\upsilon\nu B+2\eta\mu\Gamma\sigma\upsilon\nu\Gamma)=R(\eta\mu 2B+\eta\mu 2\Gamma)=$$

$$=R\cdot 2\eta\mu(B+\Gamma)\sigma\upsilon\nu(B-\Gamma)=2R\cdot\eta\mu A\sigma\upsilon\nu(B-\Gamma)=\alpha\sigma\upsilon\nu(B-\Gamma),$$

6. $$(\beta+\gamma-\alpha)\left(\sigma\varphi\frac{B}{2}+\sigma\varphi\frac{\Gamma}{2}\right)=2\alpha\sigma\varphi\frac{A}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$(\beta+\gamma-\alpha)\left(\sigma\varphi\frac{B}{2}+\sigma\varphi\frac{\Gamma}{2}\right)=2R(\eta\mu B+\eta\mu\Gamma-\eta\mu A)\cdot\frac{\eta\mu\left(\frac{B+\Gamma}{2}\right)}{\eta\mu\frac{B}{2}\cdot\eta\mu\frac{\Gamma}{2}}=$$

$$=2R\cdot 4\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{A}{2}\cdot\frac{\sigma\upsilon\nu\frac{A}{2}}{\eta\mu\frac{B}{2}\eta\mu\frac{\Gamma}{2}}=8R\sigma\upsilon\nu^2\frac{A}{2}=$$

$$=8R\cdot\frac{\eta\mu\frac{A}{2}}{\eta\mu\frac{A}{2}}\cdot\sigma\upsilon\nu^2\frac{A}{2}=4R\;2\eta\mu\frac{A}{2}\cdot\sigma\upsilon\nu\frac{A}{2}\cdot\frac{\sigma\upsilon\nu\frac{A}{2}}{\eta\mu\frac{A}{2}}=$$

$$=4R\cdot\eta\mu A\cdot\sigma\varphi\frac{A}{2}=2\cdot 2R\eta\mu A\sigma\varphi\frac{A}{2}=2\;\alpha\sigma\varphi\frac{A}{2}.$$

7. $$\Sigma\frac{\beta^2-\gamma^2}{\alpha^2}\eta\mu 2A=0.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\beta^2-\gamma^2}{\alpha^2}\eta\mu 2A=\frac{4R^2\eta\mu^2B-4R^2\eta\mu^3\Gamma}{4R^2\eta\mu^2A}\eta\mu 2A=\frac{\eta\mu^2B-\eta\mu^2\Gamma}{\eta\mu^2A}\cdot 2\eta\mu A\sigma\upsilon\nu A=$$

$$=2\frac{\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)}{\eta\mu A}\sigma\upsilon\nu A=2\cdot\frac{\eta\mu A\eta\mu(B-\Gamma)}{\eta\mu A}\cdot\sigma\upsilon\nu A=$$

$$=2\eta\mu(B-\Gamma)\sigma\upsilon\nu A=-2\eta\mu(B-\Gamma)\sigma\upsilon\nu(B+\Gamma)=2\eta\mu(\Gamma-B)\sigma\upsilon\nu(\Gamma+B)=\eta\mu 2\Gamma-\eta\mu 2B$$

καὶ διὰ κυκλικῆς ἐναλλαγῆς θὰ ἔχωμεν :

$$\Sigma\ \frac{\beta^2-\gamma^2}{\alpha^2}\ \eta\mu 2A=\eta\mu 2\Gamma-\eta\mu 2B+\eta\mu 2A-\eta\mu 2\Gamma+\eta\mu 2B-\eta\mu 2A=0.$$

122. Εἰς πᾶν τρίγωνον ΑΒΓ, νὰ ἀποδειχθῇ ὅτι :

1. $$\frac{\alpha\eta\mu(B-\Gamma)}{\beta^2-\gamma^2}=\frac{\beta\eta\mu(\Gamma-A)}{\gamma^2-\alpha^2}=\frac{\gamma\eta\mu(A-B)}{\alpha^2-\beta^2}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{\alpha\eta\mu(B-\Gamma)}{\beta^2-\gamma^2}=\frac{2R\eta\mu A\eta\mu(B-\Gamma)}{\beta^2-\gamma^2}=\frac{2R\cdot\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)}{\beta^2-\gamma^2}=$$

$$=\frac{2R(\eta\mu^2B-\eta\mu^2\Gamma)}{\beta^2-\gamma^2}=\frac{(4R^2\eta\mu^2B-4R^2\eta\mu^2\Gamma)}{2R(\beta^2-\gamma^2)}=\frac{\beta^2-\gamma^2}{2R(\beta^2-\gamma^2)}=\frac{1}{2R}.$$

Ὁμοίως ἀποδεικνύεται ὅτι καὶ τὰ ἄλλα κλάσματα ἰσοῦνται πρὸς $\frac{1}{2R}$. Ἄρα ἰσχύουν αἱ δοθεῖσαι ἰσότητες.

2. $$\Sigma\ \alpha\eta\mu\frac{B-\Gamma}{2}\ \eta\mu\ \frac{A}{2}=0.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\alpha\cdot\eta\mu\frac{B-\Gamma}{2}\ \eta\mu\ \frac{A}{2}=2R\eta\mu A\cdot\eta\mu\frac{B-\Gamma}{2}\ \eta\mu\ \frac{A}{2}=2R\cdot 2\eta\mu\frac{A}{2}\ \sigma\upsilon\nu\ \frac{A}{2}\ \eta\mu\ \frac{B-\Gamma}{2}\ \eta\mu\frac{A}{2}$$

$$=2R\cdot\eta\mu^2\ \frac{A}{2}\cdot 2\eta\mu\ \frac{B+\Gamma}{2}\ \eta\mu\ \frac{B-\Gamma}{2}=4R\eta\mu^2\frac{A}{2}\left(\eta\mu^2\frac{B}{2}-\eta\mu^2\frac{\Gamma}{2}\right).$$

Κατ' ἀκολουθίαν :

$$\Sigma\ \alpha\ \eta\mu\frac{B-\Gamma}{2}\ \eta\mu\ \frac{A}{2}=4R\Sigma\ \eta\mu^2\ \frac{A}{2}\left(\eta\mu^2\frac{B}{2}-\eta\mu^2\ \frac{\Gamma}{2}\right)=4R\cdot 0=0.$$

3. $$\Sigma\ \alpha^2\eta\mu(B-\Gamma\sigma\tau\epsilon\mu A=0.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\alpha^2\eta\mu(B-\Gamma)\sigma\tau\epsilon\mu A=4R^2\eta\mu^2A\eta\mu(B-\Gamma)\cdot\frac{1}{\eta\mu A}=4R^2\eta\mu A\ \eta\mu(B-\Gamma)=$$

$$=4R^2\cdot\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)=4R^2(\eta\mu^2B-\eta\mu^2\Gamma).$$

Κατ' ἀκολουθίαν :

$$\Sigma\,\alpha^2\eta\mu(B-\Gamma)\sigma\tau\epsilon\mu A=4R^2\cdot\Sigma(\eta\mu^2B-\eta\mu^2\Gamma)=4R^2\cdot 0=0.$$

4. $$\Sigma\ (\beta-\gamma)\sigma\varphi\ \frac{A}{2}=0.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\beta-\gamma)\sigma\varphi\ \frac{A}{2}=2R(\eta\mu B-\eta\mu\Gamma)\sigma\varphi\ \frac{A}{2}=4R\eta\mu\ \frac{B-\Gamma}{2}\ \sigma\upsilon\nu\ \frac{B+\Gamma}{2}\ \sigma\varphi\ \frac{A}{2}=$$

$$= 4R\eta\mu\frac{B-\Gamma}{2}\cdot\eta\mu\frac{A}{2}\cdot\frac{\sigma\upsilon\nu\frac{A}{2}}{\eta\mu\frac{A}{2}} = 4R\eta\mu\frac{B-\Gamma}{2}\;\sigma\upsilon\nu\frac{A}{2} =$$

$$= 4R\eta\mu\frac{B-\Gamma}{2}\;\eta\mu\frac{B+\Gamma}{2} = 2R\cdot 2\eta\mu\frac{B+\Gamma}{2}\eta\mu\frac{B-\Gamma}{2} = 2R(\sigma\upsilon\nu\Gamma-\sigma\upsilon\nu B).$$

Διὰ κυκλικῆς δ' ἐναλλαγῆς τῶν Α, Β, Γ λαμβάνομεν :

$$\Sigma(\beta-\gamma)\sigma\varphi\frac{A}{2} = 2R(\sigma\upsilon\nu\Gamma-\sigma\upsilon\nu B)+2R(\sigma\upsilon\nu A-\sigma\upsilon\nu\Gamma)+2R(\sigma\upsilon\nu B-\sigma\upsilon\nu A) = 2R\cdot 0 = 0.$$

5. $$\boldsymbol{\Sigma(\alpha-\beta)\varepsilon\varphi\frac{A+B}{2} = 0.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\alpha-\beta)\varepsilon\varphi\frac{A+B}{2} = 2R(\eta\mu A-\eta\mu B)\cdot\varepsilon\varphi\frac{A+B}{2} =$$

$$= 2R\cdot 2\eta\mu\frac{A-B}{2}\;\sigma\upsilon\nu\frac{A+B}{2}\cdot\frac{\eta\mu\frac{(A+B)}{2}}{\sigma\upsilon\nu\frac{A+B}{2}} = 4R\eta\mu\frac{A-B}{2}\;\eta\mu\frac{A+B}{2} =$$

$$= 4R\left(\eta\mu^2\frac{A}{2}-\eta\mu^2\frac{B}{2}\right).$$

Κατ' ἀκολουθίαν :

$$\Sigma(\alpha-\beta)\varepsilon\varphi\frac{A+B}{2} = 4R\cdot\Sigma\left(\eta\mu^2\frac{A}{2}-\eta\mu^2\frac{B}{2}\right) = 4R\cdot 0 = 0.$$

6. $$\boldsymbol{\Sigma(\alpha+\beta)\varepsilon\varphi\frac{A-B}{2} = 0.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$(\alpha+\beta)\varepsilon\varphi\frac{A-B}{2} = 2R(\eta\mu A+\eta\mu B)\varepsilon\varphi\frac{A-B}{2} = 4R\eta\mu\frac{A+B}{2}\;\sigma\upsilon\nu\frac{A-B}{2}\;\varepsilon\varphi\frac{A-B}{2} =$$

$$= 4R\eta\mu\frac{A+B}{2}\;\eta\mu\frac{A-B}{2} = 4R\left(\eta\mu^2\frac{A}{2}-\eta\mu^2\frac{B}{2}\right).$$

Κατ' άκολουθιαν :

$$\Sigma(\alpha+\beta)\varepsilon\varphi\frac{A-B}{2} = 4R\cdot\Sigma\left(\eta\mu^2\frac{A}{2}-\eta\mu^2\frac{B}{2}\right) = 4R\;0 = 0.$$

7. $$\boldsymbol{\Sigma\frac{\alpha^2\eta\mu(B-\Gamma)}{\eta\mu B+\eta\mu\Gamma} = 0.}$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{\alpha^2\eta\mu(B-\Gamma)}{\eta\mu B+\eta\mu\Gamma} = \frac{4R^2\eta\mu^2 A\eta\mu(B-\Gamma)}{\eta\mu B+\eta\mu\Gamma} = \frac{4R^2\cdot\eta\mu A\cdot\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)}{\eta\mu B+\eta\mu\Gamma} =$$

$$= \frac{4R^2 \eta\mu A(\eta\mu^2 B - \eta\mu^2 \Gamma)}{\eta\mu B + \eta\mu \Gamma} = 4R^2 \eta\mu A(\eta\mu B - \eta\mu \Gamma).$$

Κατ' ἀκολουθίαν:

$$\Sigma \frac{\alpha^2 \eta\mu(B-\Gamma)}{\eta\mu B + \eta\mu \Gamma} = 4R^2 \Sigma \eta\mu A(\eta\mu B - \eta\mu \Gamma) = 4R^2 \cdot 0 = 0.$$

8. $\boldsymbol{\Sigma\ \alpha^2(\sigma\upsilon\nu^2 B - \sigma\upsilon\nu^2 \Gamma) = 0.}$

Λύσις. Ἔχομεν: $\alpha^2(\sigma\upsilon\nu^2 B - \sigma\upsilon\nu^2 \Gamma) = 4R^2 \eta\mu^2 A(\eta\mu^2 \Gamma - \eta\mu^2 B).$

Κατ' ἀκολουθίαν:

$$\Sigma \alpha^2(\sigma\upsilon\nu^2 B - \sigma\upsilon\nu^2 \Gamma) = 4R^2 \Sigma \eta\mu^2 A(\eta\mu^2 \Gamma - \eta\mu^2 B) = 4R^2 \cdot 0 = 0.$$

9. $\boldsymbol{\Sigma\ (\beta^2 - \gamma^2)\sigma\varphi A = 0.}$

Λύσις. Ἔχομεν διαδοχικῶς:

$$(\beta^2 - \gamma^2)\sigma\varphi A = 4R^2(\eta\mu^2 B - \eta\mu^2 \Gamma)\sigma\varphi A = 4R^2 \eta\mu(B+\Gamma)\eta\mu(B-\Gamma)\sigma\varphi A =$$

$$= 4R^2 \eta\mu A \cdot \eta\mu(B-\Gamma) \cdot \frac{\sigma\upsilon\nu A}{\eta\mu A} = 4R^2 \eta\mu(B-\Gamma)\sigma\upsilon\nu A =$$

$$= -4R^2 \eta\mu(B-\Gamma)\sigma\upsilon\nu(\Gamma+B) = 2R^2 \cdot 2\eta\mu(\Gamma-B)\sigma\upsilon\nu(\Gamma+B) =$$

$$= 2R^2 \cdot (\eta\mu 2\Gamma - \eta\mu 2B).$$

Κατ' ἀκολουθίαν:

$$\Sigma(\beta^2 - \gamma^2)\sigma\varphi A = 2R^2 \Sigma(\eta\mu 2\Gamma - \eta\mu 2B) = 2R^2 \cdot 0 = 0.$$

10. $\boldsymbol{\Sigma\ \alpha \cdot \eta\mu \frac{B-\Gamma}{2} \sigma\tau\epsilon\mu \frac{A}{2} = 0.}$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\alpha\ \eta\mu \frac{B-\Gamma}{2} \sigma\tau\epsilon\mu \frac{A}{2} = 2R\eta\mu A \eta\mu \frac{B-\Gamma}{2} \sigma\tau\epsilon\mu \frac{A}{2} =$$

$$= 4R\eta\mu \frac{A}{2} \sigma\upsilon\nu \frac{A}{2} \eta\mu \frac{B-\Gamma}{2} \cdot \frac{1}{\eta\mu \frac{A}{2}} =$$

$$= 4R\sigma\upsilon\nu \frac{A}{2} \eta\mu \frac{B-\Gamma}{2} = 4R\eta\mu \frac{B+\Gamma}{2} \eta\mu \frac{B-\Gamma}{2} =$$

$$= 4R\left(\eta\mu^2 \frac{B}{2} - \eta\mu^2 \frac{\Gamma}{2}\right).$$

Κατ' ἀκολουθίαν:

$$\Sigma\alpha\eta\mu \frac{B-\Gamma}{2} \sigma\tau\epsilon\mu \frac{A}{2} = 4R \cdot \Sigma\left(\eta\mu^2 \frac{B}{2} - \eta\mu^2 \frac{\Gamma}{2}\right) = 4R \cdot 0 = 0.$$

11. $$\Sigma\ \alpha\eta\mu\frac{B-\Gamma}{2}\eta\mu\frac{A}{2}=0.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\alpha\,\eta\mu\frac{B-\Gamma}{2}\,\eta\mu\frac{A}{2}=2R\eta\mu A\eta\mu\frac{B-\Gamma}{2}\,\eta\mu\frac{A}{2}=$$

$$=4R\eta\mu\frac{A}{2}\,\sigma\upsilon\nu\frac{A}{2}\,\eta\mu\frac{B-\Gamma}{2}\,\eta\mu\frac{A}{2}=4R\eta\mu^2\frac{A}{2}\cdot\eta\mu\frac{B+\Gamma}{2}\,\eta\mu\frac{B-\Gamma}{2}=$$

$$=4R\eta\mu^2\frac{A}{2}\left(\eta\mu^2\frac{B}{2}-\eta\mu^2\frac{\Gamma}{2}\right).$$

Κατ' ἀκολουθίαν:

$$\Sigma\alpha\eta\mu\frac{B-\Gamma}{2}\,\eta\mu\frac{A}{2}=4R\Sigma\eta\mu^2\frac{A}{2}\left(\eta\mu^2\frac{B}{2}-\eta\mu^2\frac{\Gamma}{2}\right)=4R\cdot 0=0.$$

12, $$\Sigma\ \frac{\beta}{\alpha\eta\mu\Gamma}=2\Sigma\sigma\varphi A.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\beta}{\alpha\eta\mu\Gamma}=\frac{\eta\mu B}{\eta\mu A\eta\mu\Gamma}=\frac{\eta\mu(A+\Gamma)}{\eta\mu A\eta\mu\Gamma}=\frac{\eta\mu A\sigma\upsilon\nu\Gamma+\eta\mu\Gamma\sigma\upsilon\nu A}{\eta\mu A\eta\mu\Gamma}=\sigma\varphi\Gamma+\sigma\varphi A.$$

Κατ' ἀκολουθίαν:

$$\Sigma\frac{\beta}{\alpha\eta\mu\Gamma}=\Sigma(\sigma\varphi\Gamma+\sigma\varphi A)=(\sigma\varphi\Gamma+\sigma\varphi A)+(\sigma\varphi A+\sigma\varphi B)+(\sigma\varphi B+\sigma\varphi\Gamma)=2\Sigma\sigma\varphi A$$

13. $$\Sigma\ \frac{\sigma\upsilon\nu A\sigma\upsilon\nu B}{\alpha\beta}=\frac{1}{4R^2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{\sigma\upsilon\nu A\sigma\upsilon\nu B}{\alpha\beta}=\frac{\sigma\upsilon\nu A\sigma\upsilon\nu B}{4R^2\eta\mu A\eta\mu B}=\frac{1}{4R^2}\cdot\sigma\varphi A\sigma\varphi B.$$

Κατ' ἀκολουθίαν:

$$\Sigma\frac{\sigma\upsilon\nu A\sigma\upsilon\nu B}{\alpha\beta}=\frac{1}{4R^2}\cdot(\sigma\varphi A\sigma\varphi B+\sigma\varphi B\sigma\varphi\Gamma+\sigma\varphi\Gamma\sigma\varphi A)=\frac{1}{4R^2}\cdot 1=\frac{1}{4R^2}.$$

14. $$\Sigma\ \frac{1}{\alpha}\sigma\upsilon\nu^2\frac{A}{2}=\frac{\tau^2}{\alpha\beta\gamma}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{1}{\alpha}\cdot\sigma\upsilon\nu^2\frac{A}{2}=\frac{1}{\alpha}\cdot\frac{\tau(\tau-\alpha)}{\beta\gamma}=\frac{\tau^2-\alpha\tau}{\alpha\beta\gamma}.$$

Κατ' ἀκολουθίαν:

$$\Sigma\frac{1}{\alpha}\cdot\sigma\upsilon\nu^2\frac{A}{2}=\frac{\tau^2-\alpha\tau}{\alpha\beta\gamma}+\frac{\tau^2-\beta\tau}{\alpha\beta\gamma}+\frac{\tau^2-\gamma\tau}{\alpha\beta\gamma}=\frac{3\tau^2-(\alpha+\beta+\gamma)\tau}{\alpha\beta\gamma}=$$

$$=\frac{3\tau^2-2\tau^2}{\alpha\beta\gamma}=\frac{\tau^2}{\alpha\beta\gamma}.$$

15. $$\Sigma \cdot \frac{1}{\alpha} \cdot \eta\mu^2 \frac{A}{2} = \frac{2\Sigma\alpha\beta - \Sigma\alpha^2}{4\alpha\beta\gamma}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\frac{1}{\alpha}\,\eta\mu^2\frac{A}{2} = \frac{1}{\alpha}\cdot\frac{(\tau-\beta)(\tau-\gamma)}{\beta\gamma} = \frac{\tau^2-\beta\tau-\gamma\tau+\beta\gamma}{\alpha\beta\gamma}.$$

Κατ᾽ ἀκολουθίαν:

$$\Sigma\frac{1}{\alpha}\cdot\eta\mu^2\frac{A}{2} = \frac{\tau^2-\beta\tau-\gamma\tau+\beta\gamma}{\alpha\beta\gamma} + \frac{\tau^2-\gamma\tau-\alpha\tau+\gamma\alpha}{\alpha\beta\gamma} + \frac{\tau^2-\alpha\tau-\beta\tau+\alpha\beta}{\alpha\beta\gamma} =$$

$$= \frac{3\tau^2-2\tau(\alpha+\beta+\gamma)+\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} = \frac{-\tau^2+\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} =$$

$$= \frac{-\left(\frac{\alpha+\beta+\gamma}{2}\right)^2+\alpha\beta+\beta\gamma+\gamma\alpha}{\alpha\beta\gamma} =$$

$$= \frac{-\alpha^2-\beta^2-\gamma^2-2\alpha\beta-2\beta\gamma-2\gamma\alpha+4\alpha\beta+4\beta\gamma+4\gamma\alpha}{4\alpha\beta\gamma} = \frac{2\Sigma\alpha\beta-\Sigma\alpha^2}{4\alpha\beta\gamma}.$$

16. $$\Sigma\,\sigma\varphi\frac{A}{2} = \frac{\tau}{\tau-\alpha}\,\sigma\varphi\frac{A}{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\Sigma\,\sigma\varphi\frac{A}{2} = \sigma\varphi\frac{A}{2} + \sigma\varphi\frac{B}{2} + \sigma\varphi\frac{\Gamma}{2} =$$

$$= \sqrt{\frac{\tau(\tau-\alpha)}{(\tau-\beta)(\tau-\gamma)}} + \sqrt{\frac{\tau(\tau-\beta)}{(\tau-\gamma)(\tau-\alpha)}} + \sqrt{\frac{\tau(\tau-\gamma)}{(\tau-\alpha)(\tau-\beta)}} =$$

$$= \frac{E}{(\tau-\beta)(\tau-\gamma)} + \frac{E}{(\tau-\gamma)(\tau-\alpha)} + \frac{E}{(\tau-\alpha)(\tau-\beta)} =$$

$$= E\cdot\frac{\tau-\alpha+\tau-\beta+\tau-\gamma}{(\tau-\alpha)(\tau-\beta)(\tau-\gamma)} = \frac{\tau\cdot E}{(\tau-\alpha)(\tau-\beta)(\tau-\gamma)} =$$

$$= \frac{\tau}{\tau-\alpha}\cdot\frac{E}{(\tau-\beta)(\tau-\gamma)} = \frac{\tau}{\tau-\alpha}\cdot\frac{\sqrt{\tau(\tau-\alpha)(\tau-\beta)(\tau-\gamma)}}{(\tau-\beta)(\tau-\gamma)} =$$

$$= \frac{\tau}{\tau-\alpha}\cdot\sqrt{\frac{\tau(\tau-\alpha)}{(\tau-\beta)(\tau-\gamma)}} = \frac{\tau}{\tau-\alpha}\,\sigma\varphi\frac{A}{2}.$$

17. $$\Sigma\,\alpha\sigma\upsilon\nu A = \frac{2E}{R}.$$

Λύσις. Ἔχομσν διαδοχικῶς:

$$\Sigma\,\alpha\,\sigma\upsilon\nu A = \alpha\sigma\upsilon\nu A + \beta\sigma\upsilon\nu B + \gamma\sigma\upsilon\nu\Gamma = 2R(\eta\mu A\sigma\upsilon\nu A + \eta\mu B\sigma\upsilon\nu B + \eta\mu\Gamma\sigma\upsilon\nu\Gamma) =$$

$$= R(\eta\mu 2A + \eta\mu 2B + \eta\mu 2\Gamma)$$

$$=4R\eta\mu A\eta\mu B\eta\mu\Gamma=4R\cdot\frac{\alpha}{2R}\cdot\frac{\beta}{2R}\cdot\frac{\gamma}{2R}=\frac{\alpha\beta\gamma}{2R^2}=\frac{4ER}{2R^2}=\frac{2E}{R}.$$

18. $$\frac{(\alpha+\beta+\gamma)^2}{\alpha^2+\beta^2+\gamma^2}=\frac{\Sigma\,\sigma\varphi\frac{A}{2}}{\Sigma\,\sigma\varphi A}.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\frac{(\alpha+\beta+\gamma)^2}{\alpha^2+\beta^2+\gamma^2}=\frac{(\eta\mu A+\eta\mu B+\eta\mu\Gamma)^2}{\eta\mu^2A+\eta\mu^2B+\eta\mu^2\Gamma}=\frac{16\sigma\upsilon\nu^2\frac{A}{2}\sigma\upsilon\nu^2\frac{B}{2}\sigma\upsilon\nu^2\frac{\Gamma}{2}}{\eta\mu^2A+\eta\mu^2B+\eta\mu^2\Gamma}=$$

$$=2\sigma\varphi\frac{A}{2}\sigma\varphi\frac{B}{2}\sigma\varphi\frac{\Gamma}{2}\cdot\frac{\eta\mu A\eta\mu B\eta\mu\Gamma}{\eta\mu^2A+\eta\mu^2B+\eta\mu^2\Gamma}$$

$$=2\sigma\varphi\frac{A}{2}\sigma\varphi\frac{B}{2}\sigma\varphi\frac{\Gamma}{2}\cdot\frac{\eta\mu^2A+\eta\mu^2B+\eta\mu^2\Gamma}{\eta\mu A\eta\mu B\eta\mu\Gamma}$$

$$=2\sigma\varphi\frac{A}{2}\sigma\varphi\frac{B}{2}\sigma\varphi\frac{\Gamma}{2}\cdot\left(\frac{\eta\mu A}{\eta\mu B\eta\mu\Gamma}+\frac{\eta\mu B}{\eta\mu\Gamma\eta\mu A}+\frac{\eta\mu\Gamma}{\eta\mu A\eta\mu B}\right)$$

$$=2\sigma\varphi\frac{A}{2}\sigma\varphi\frac{B}{2}\sigma\varphi\frac{\Gamma}{2}\cdot\left[\frac{\eta\mu(B+\Gamma)}{\eta\mu B\eta\mu\Gamma}+\frac{\eta\mu(\Gamma+A)}{\eta\mu\Gamma\eta\mu A}+\frac{\eta\mu(A+B)}{\eta\mu A\eta\mu B}\right]$$

$$=2\sigma\varphi\frac{A}{2}\sigma\varphi\frac{B}{2}\sigma\varphi\frac{\Gamma}{2}\cdot[\sigma\varphi\Gamma+\sigma\varphi B+\sigma\varphi A+\sigma\varphi\Gamma+\sigma\varphi B+\sigma\varphi A]$$

$$=\frac{\sigma\varphi\frac{A}{2}\sigma\varphi\frac{B}{2}\sigma\varphi\frac{\Gamma}{2}}{\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma}=\frac{\sigma\varphi\frac{A}{2}+\sigma\varphi\frac{B}{2}+\sigma\varphi\frac{\Gamma}{2}}{\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma}=\frac{\Sigma\,\sigma\varphi\frac{A}{2}}{\Sigma\,\sigma\varphi A}.$$

19. $$\Sigma\,\beta\gamma\sigma\upsilon\nu^2\frac{A}{2}=\tau^2.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\beta\gamma\sigma\upsilon\nu^2\frac{A}{2}=\beta\gamma\,\frac{\tau(\tau-\alpha)}{\beta\gamma}=\tau(\tau-\alpha)=\tau^2-\alpha\tau.$$

Κατ' ἀκολουθίαν :

$$\Sigma\,\beta\gamma\sigma\upsilon\nu^2\frac{A}{2}=\Sigma\,\tau(\tau-\alpha)=\tau^2-\alpha\tau+\tau^2-\beta\tau+\tau^2-\beta\tau+\tau^2-\gamma\tau=$$

$$=3\tau^2-\tau(\alpha+\beta+\gamma)=3\tau^2-\tau\cdot2\tau=3\tau^2-2\tau^2=\tau^2.$$

123. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $$\alpha^2=\beta^2+\gamma^2-4E\sigma\varphi A.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\alpha^2=\beta^2+\gamma^2-2\beta\gamma\sigma\upsilon\nu A=\beta^2+\gamma^2-2\cdot\frac{2E}{\eta\mu A}\cdot\sigma\upsilon\nu A=\beta^2+\gamma^2-4E\sigma\varphi A.$$

2. $$2E(\sigma\varphi B-\sigma\varphi A)=\alpha^2-\beta^2.$$

Λύσις. Ἔχομεν διαδοχικῶς :

$$-\sigma\varphi A=\frac{\alpha^2-\beta^2-\gamma^2}{4E}\ \text{καὶ}\ -\sigma\varphi B=\frac{\beta^2-\gamma^2-\alpha^2}{4E}\quad\text{ἢ}\quad\sigma\varphi B=\frac{\alpha^2+\gamma^2-\beta^2}{4}.$$

Ἄρα $\sigma\varphi B - \sigma\varphi A = \dfrac{\alpha^2+\gamma^2-\beta^2}{4E} + \dfrac{\alpha^2-\beta^2-\gamma^2}{4E} = \dfrac{2(\alpha^2-\beta^2)}{4E} = \dfrac{\alpha^2-\beta^0}{2E}$

ἐξ οὗ $$2E(\sigma\varphi B - \sigma\varphi A) = \alpha^2 - \beta^2.$$

3. $$\boldsymbol{\alpha^2+\beta^2+\gamma^2 = 4E \cdot \Sigma\sigma\varphi A.}$$

Λύσις. Ἐκ τῆς ἀνωτέρω ἀσκήσεως (1) λαμβάνομεν:

$$4E\sigma\varphi A = \beta^2+\gamma^2-\alpha^2$$

καὶ ὁμοίως: $4E\sigma\varphi B = \gamma^2+\alpha^2-\beta^2$, $4E\sigma\varphi \Gamma = \alpha^2+\beta^2-\gamma^2$,

ὁπότε: $4E(\sigma\varphi A + \sigma\varphi B + \sigma\varphi \Gamma) = \alpha^2+\beta^2+\gamma^2.$

4. $$\boldsymbol{1 - \varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2} = \frac{\gamma}{\tau}.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$1-\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{B}{2} = 1 - \frac{\eta\mu\frac{A}{2}\,\eta\mu\frac{B}{2}}{\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}} = \frac{\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2} - \eta\mu\frac{A}{2}\,\eta\mu\frac{B}{2}}{\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}} =$$

$$= \frac{\sigma\upsilon\nu\left(\frac{A}{2}+\frac{B}{2}\right)}{\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}} = \frac{\eta\mu\frac{\Gamma}{2}}{\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}} = \frac{4\eta\mu\frac{\Gamma}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}}{4\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}} =$$

$$= \frac{2\cdot\eta\mu\Gamma}{\eta\mu A+\eta\mu B+\eta\mu\Gamma} = \frac{2\cdot 2R\eta\mu\Gamma}{2R\eta\mu A+2R\eta\mu B+2R\eta\mu\Gamma} = \frac{2\cdot\gamma}{\alpha+\beta+\gamma} = \frac{2\gamma}{2\tau} = \frac{\gamma}{\tau}.$$

124. Τρίγωνον ΑΒΓ εἶναι ἰσοσκελές, ὅταν ἰσχύουν αἱ σχέσεις:

1. $$\boldsymbol{\alpha = 2\beta\eta\mu\frac{A}{2}.}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$\alpha = 2\beta\eta\mu\frac{A}{2} \quad \text{ἢ} \quad 2R\eta\mu A = 2\cdot 2R\eta\mu B\eta\mu\frac{A}{2} \quad \text{ἢ}$$

$$2\eta\mu\frac{A}{2}\,\sigma\upsilon\nu\frac{A}{2} = 2\cdot 2\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\eta\mu\frac{A}{2} \quad \text{ἢ} \quad \sigma\upsilon\nu\frac{A}{2} = 2\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{B}{2} = \eta\mu B$$

$$\text{ἢ} \quad \sigma\upsilon\nu\frac{A}{2} = \sigma\upsilon\nu(90^0 - B) \Longrightarrow \frac{A}{2} = 90^0 - B = \frac{A}{2}+\frac{B}{2}+\frac{\Gamma}{2}-\frac{B}{2}-\frac{B}{2}$$

ἐξ οὗ: $$B = \Gamma.$$

2. $$\boldsymbol{\eta\mu A = 2\eta\mu B\sigma\upsilon\nu\Gamma.}$$

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$\eta\mu(B+\Gamma) = 2\eta\mu B\sigma\upsilon\nu\Gamma \quad \text{ἢ} \quad \eta\mu B\sigma\upsilon\nu\Gamma + \eta\mu\Gamma\sigma\upsilon\nu B = 2\eta\mu B\sigma\upsilon\nu\Gamma$$

ἢ $$\eta\mu B\sigma\upsilon\nu\Gamma - \eta\mu\Gamma\sigma\upsilon\nu B = 0 \quad \text{ἢ} \quad \eta\mu(B-\Gamma) = 0 = \eta\mu 0^0.$$

Ἄρα $$B - \Gamma = 0 \quad \text{ἢ} \quad B = \Gamma.$$

3, $\alpha = 2\beta \sigma \upsilon \nu \Gamma$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$2R\eta\mu A = 2 \cdot 2R\eta\mu B\sigma\upsilon\nu\Gamma \quad \text{ἢ} \quad \eta\mu A = 2\eta\mu B\sigma\upsilon\nu\Gamma$$

ἤ, λόγῳ τῆς προηγουμένης ἀσκήσεως, $B = \Gamma$.

4. $(\tau - \beta)\sigma\varphi \dfrac{\Gamma}{2} = \tau\varepsilon\varphi \dfrac{B}{2}$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$(\tau-\beta) \cdot \sqrt{\frac{\tau(\tau-\gamma)}{(\tau-\alpha)(\tau-\beta)}} = \tau \cdot \sqrt{\frac{(\tau-\alpha)(\tau-\gamma)}{\tau(\tau-\beta)}}$$

ἢ $$(\tau-\beta)^2 \cdot \frac{\tau(\tau-\gamma)}{(\tau-\alpha)(\tau-\beta)} = \tau^2 \cdot \frac{(\tau-\alpha)(\tau-\gamma)}{\tau(\tau-\beta)}$$

ἢ $$(\tau-\beta)^2 = (\tau-\alpha)^2 \quad \text{ἢ} \quad \tau-\beta = \tau-\alpha \implies \alpha = \beta.$$

5. $2\upsilon_1 = \alpha\sigma\varphi \dfrac{A}{2}$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$2 \cdot \beta\eta\mu\Gamma = \alpha\sigma\varphi \frac{A}{2} \quad \text{ἢ} \quad 4R\eta\mu B\eta\mu\Gamma = 2R\eta\mu A\sigma\varphi \frac{A}{2}$$

ἢ $$2\eta\mu B\eta\mu\Gamma = 2 \cdot \eta\mu \frac{A}{2} \sigma\upsilon\nu \frac{A}{2} \cdot \frac{\sigma\upsilon\nu \frac{A}{2}}{\eta\mu \frac{A}{2}} = 2\sigma\upsilon\nu^2 \frac{A}{2}$$

ἢ $$\sigma\upsilon\nu(B-\Gamma) - \sigma\upsilon\nu(B+\Gamma) = 1 + \sigma\upsilon\nu A$$

ἢ $$\sigma\upsilon\nu(B-\Gamma) - \sigma\upsilon\nu(B+\Gamma) = 1 - \sigma\upsilon\nu(B+\Gamma) \quad \text{ἢ} \quad \sigma\upsilon\nu(B-\Gamma) = 1.$$

Ἄρα $$B - \Gamma = 0 \iff B = \Gamma.$$

6. $4E = \alpha^2\sigma\varphi \dfrac{A}{2}$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$4 \cdot \dfrac{1}{2}\alpha\upsilon_1 = \alpha^2 \cdot \sigma\varphi \dfrac{A}{2}$ ἢ $2\upsilon_1 = \alpha\sigma\varphi \dfrac{A}{2}$, ἐξ οὗ βάσει, τῆς προηγουμένης ἀσκήσεως, εἶναι: $B = \Gamma$.

7, $\dfrac{\Sigma\alpha^2}{2E} = \sigma\varphi \dfrac{A}{2} + 3\varepsilon\varphi \dfrac{A}{2}$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$\frac{\alpha^2}{2E} + \frac{\beta^2}{2E} + \frac{\gamma^2}{2E} = \sigma\varphi \frac{A}{2} + 3\varepsilon\varphi \frac{A}{2} \quad \text{ἢ} \quad \frac{\alpha^2}{\alpha\upsilon_1} + \frac{\beta^2}{\beta\upsilon_2} + \frac{\gamma^2}{\gamma\upsilon_3} = \sigma\varphi \frac{A}{2} + 3\varepsilon\varphi \frac{A}{2}$$

ἢ $$\frac{\alpha}{\upsilon_1} + \frac{\beta}{\upsilon_2} + \frac{\gamma}{\upsilon_3} = \sigma\varphi \frac{A}{2} + 3\varepsilon\varphi \frac{A}{2} \qquad (1)$$

Ἄγομεν τὸ ὕψος $AH=\upsilon_1$ καὶ θὰ ἔχωμεν:

$$\alpha=BH+H\Gamma=\upsilon_1(\sigma\varphi B+\sigma\varphi\Gamma) \quad \text{ἢ} \quad \frac{\alpha}{\upsilon_1}=\sigma\varphi B+\sigma\varphi\Gamma,$$

Ἄρα ἡ (1) γράφεται:

$$2(\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma)=\sigma\varphi\frac{A}{2}+3\varepsilon\varphi\frac{A}{2}$$

ἐξ οὗ:

$$2(\sigma\varphi B+\sigma\varphi\Gamma)=\sigma\varphi\frac{A}{2}+3\varepsilon\varphi\frac{A}{2}-2\sigma\varphi A$$

ἢ

$$\frac{2\eta\mu(B+\Gamma)}{\eta\mu B\eta\mu\Gamma}=\frac{1+3\varepsilon\varphi^2\frac{A}{2}}{\varepsilon\varphi\frac{A}{2}}-\frac{1-\varepsilon\varphi^2\frac{A}{2}}{\varepsilon\varphi\frac{A}{2}}=4\varepsilon\varphi\frac{A}{2}$$

ἢ

$$\frac{2\eta\mu A}{\eta\mu B\eta\mu\Gamma}=4\varepsilon\varphi\frac{A}{2} \quad \text{ἢ} \quad \eta\mu A=4\varepsilon\varphi\frac{A}{2}\eta\mu B\eta\mu\Gamma$$

ἢ $\eta\mu B\eta\mu\Gamma=\sigma\upsilon\nu^2\frac{A}{2}$ ἤ, λόγῳ τῆς (5), $\sigma\upsilon\nu(B-\Gamma)=1 \Longrightarrow B=\Gamma.$

8. $$\boldsymbol{\alpha\varepsilon\varphi A+\beta\varepsilon\varphi B=(\alpha+\beta)\varepsilon\varphi\frac{A+B}{2}}.$$

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$\alpha\left(\varepsilon\varphi A-\varepsilon\varphi\frac{A+B}{2}\right)=\beta\left(\varepsilon\varphi\frac{A+B}{2}-\varepsilon\varphi B\right)$$

ἢ

$$\eta\mu A\cdot\frac{\eta\mu\left(A-\frac{A+B}{2}\right)}{\sigma\upsilon\nu A\cdot\sigma\upsilon\nu\frac{A+B}{2}}=\eta\mu B\cdot\frac{\eta\mu\left(\frac{A+B}{2}-B\right)}{\sigma\upsilon\nu\frac{A+B}{2}\sigma\upsilon\nu B}$$

ἢ

$$\eta\mu A\sigma\upsilon\nu B\eta\mu\left(\frac{A-B}{2}\right)=\eta\mu B\sigma\upsilon\nu A\eta\mu\left(\frac{A-B}{2}\right)$$

ἢ

$$\eta\mu A\sigma\upsilon\nu B=\eta\mu B\sigma\upsilon\nu A$$

ἢ

$$\eta\mu A\sigma\upsilon\nu B-\eta\mu B\sigma\upsilon\nu A=0 \quad \text{ἢ} \quad \eta\mu(A-B)=0$$

ἐξ οὗ

$$A-B=0 \Longleftrightarrow A=B.$$

125. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι:

$$\boldsymbol{\eta\mu\Gamma(\sigma\upsilon\nu A+2\sigma\upsilon\nu\Gamma)=\eta\mu B(\sigma\upsilon\nu A+2\sigma\upsilon\nu B)},$$

νὰ ἀποδειχθῇ ὅτι τοῦτο εἶναι ἰσοσκελὲς ἢ ὀρθογώνιον.

Λύσις Ἡ δοθεῖσα σχέσις γράφεται:

$$\eta\mu\Gamma\sigma\upsilon\nu A+2\eta\mu\Gamma\sigma\upsilon\nu\Gamma=\eta\mu B\sigma\upsilon\nu A+2\eta\mu B\sigma\upsilon\nu B$$

ἢ

$$\eta\mu\Gamma\sigma\upsilon\nu A+\eta\mu 2\Gamma=\eta\mu B\sigma\upsilon\nu A+\eta\mu 2B$$

ἢ

$$\sigma\upsilon\nu A(\eta\mu\Gamma-\eta\mu B)=\eta\mu 2B-\eta\mu 2\Gamma=2\eta\mu(B-\Gamma)\sigma\upsilon\nu(B+\Gamma)$$
$$=-2\eta\mu(B-\Gamma)\sigma\upsilon\nu A$$

ἢ

$$\sigma\upsilon\nu A[\eta\mu\Gamma-\eta\mu B+2\eta\mu(B-\Gamma)]=0,$$

$$\text{Άρα} \quad \text{ἢ} \quad \sigma\upsilon\nu A = 0 \Longrightarrow A = 90^0$$

$$\text{ἢ} \quad \eta\mu\Gamma - \eta\mu B = -2\eta\mu(B-\Gamma) = 2\eta\mu(\Gamma - B)$$

$$\text{ἢ} \quad 2\eta\mu\frac{\Gamma-B}{2}\cdot\sigma\upsilon\nu\frac{\Gamma+B}{2} - 4\eta\mu\frac{\Gamma-B}{2}\sigma\upsilon\nu\frac{\Gamma-B}{2}$$

$$\text{ἢ} \quad \eta\mu\frac{\Gamma-B}{2}\left[\sigma\upsilon\nu\frac{\Gamma+B}{2} - 2\sigma\upsilon\nu\frac{\Gamma-B}{2}\right] = 0.$$

$$\text{Ὅθεν} \quad \text{ἢ} \quad \eta\mu\frac{\Gamma-B}{2} = 0 \Longrightarrow \Gamma = B$$

$$\text{ἢ} \quad \sigma\upsilon\nu\left(\frac{\Gamma}{2}+\frac{B}{2}\right) - 2\sigma\upsilon\nu\left(\frac{\Gamma}{2}-\frac{B}{2}\right) = 0$$

$$\text{ἢ} \quad \sigma\upsilon\nu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{B}{2} - \eta\mu\frac{\Gamma}{2}\eta\mu\frac{B}{2} - 2\sigma\upsilon\nu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{B}{2} - 2\eta\mu\frac{\Gamma}{2}\eta\mu\frac{B}{2}$$

$$\text{ἢ} \quad \sigma\upsilon\nu\frac{\Gamma}{2}\sigma\upsilon\nu\frac{B}{2} + \eta\mu\frac{\Gamma}{2}\eta\mu\frac{B}{2} = 0$$

$$\text{ἢ} \quad \sigma\upsilon\nu\frac{\Gamma-B}{2} = 0 \Longrightarrow \frac{\Gamma-B}{2} = 90^0 \Longrightarrow \Gamma - B = 180^0, \text{ ὅπερ ἄτοπον.}$$

Ἄρα ἢ $A = 90^0$ ἢ $\Gamma = B$.

126. Εἰς τρίγωνον ΑΒΓ εἶναι: $(1-\sigma\varphi\Gamma)[1+\sigma\varphi(45^0-B)]=2$. Νὰ ἀποδειχθῇ ὅτι τοῦτο εἶναι ὀρθογώνιον.

Λύσις. Ἐπειδή:

$$1+\sigma\varphi(45^0-B) = 1+\varepsilon\varphi(45^0+B) = 1+\frac{1+\varepsilon\varphi B}{1-\varepsilon\varphi B} = \frac{2}{1-\varepsilon\varphi B},$$

τότε ἡ δοθεῖσα σχέσις γράφεται:

$$(1-\sigma\varphi\Gamma)\cdot\frac{2}{1-\varepsilon\varphi B} = 2 \quad \text{ἢ} \quad 1-\sigma\varphi\Gamma = 1-\varepsilon\varphi B \quad \text{ἢ} \quad \varepsilon\varphi B = \sigma\varphi\Gamma$$

καὶ κατ' ἀκολουθίαν $B+\Gamma=90^0 \Longrightarrow A=90^0$.

127. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι: $A=90^0$ *καὶ* $4E=\alpha^2$, τὸ τρίγωνον τοῦτο θὰ εἶναι ἰσοσκελές.

Λύσις. Ἐπειδὴ $A=90^0 \Longrightarrow \beta^2+\gamma^2=\alpha^2$ καὶ $E=\frac{1}{2}\beta\gamma$,

Ἄρα ἡ σχέσις $4E=\alpha^2$ γίνεται:

$$4\cdot\frac{1}{2}\beta\gamma = \beta^2+\gamma^2 \quad \text{ἢ} \quad \beta^2+\gamma^2-2\beta\gamma=0 \quad \text{ἢ} \quad (\beta-\gamma)^2=0 \Longrightarrow \beta=\gamma.$$

128. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι:

$$\boldsymbol{\beta^3+\gamma^3-\alpha^3=\alpha^2(\beta+\gamma-\alpha) \text{ καὶ } 4\eta\mu B\eta\mu\Gamma=3,}$$

τὸ τρίγωνον τοῦτο εἶναι ἰσόπλευρον.

Λύσις. Ἡ πρώτη σχέσις γράφεται:

$$\beta^3+\gamma^3=\alpha^2(\beta+\gamma) \quad \text{ἢ} \quad (\beta+\gamma)(\beta^2-\beta\gamma+\gamma^2)=\alpha^2(\beta+\gamma)$$

$$\text{ἢ} \quad \beta^2-\beta\gamma+\gamma^2=\alpha^2$$

ἢ $\beta^2-\beta\gamma+\gamma^2=\beta^2+\gamma^2-2\beta\gamma\sigma\upsilon\nu A$, ἐξ οὗ : $\sigma\upsilon\nu A=\frac{1}{2}=\sigma\upsilon\nu 60^0$, ἢ $A=6\theta^0$.

Ἄρα $B+\Gamma=120^0$.

Ἡ δευτέρα δοθεῖσα σχέσις γράφεται :

$$2\eta\mu B\eta\mu\Gamma=\frac{3}{2} \quad \text{ἢ} \quad \sigma\upsilon\nu(B-\Gamma)-\sigma\upsilon\nu(B+\Gamma)=\frac{3}{2}$$

ἢ $\sigma\upsilon\nu(B-\Gamma)-\sigma\upsilon\nu 120^0=\frac{3}{2}$ ἢ $\sigma\upsilon\nu(B-\Gamma)+\frac{1}{2}=\frac{3}{2}$ ἢ $\sigma\upsilon\nu(B-\Gamma)=1$

ἢ $\sigma\upsilon\nu(B-\Gamma)=\sigma\upsilon\nu\ 0 \Longrightarrow B-\Gamma\ 0 \Longrightarrow B=\Gamma=60^0$.

129. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $A=120^0$, νὰ ἀποδειχθῇ ὅτι :

$$\gamma(\alpha^2-\gamma^2)=\beta(\alpha^2-\beta^2).$$

Λύσις. Γνωρίζομεν ὅτι : $\alpha^2=\beta^2+\gamma^2-2\beta\gamma\sigma\upsilon\nu A$

ἢ $\alpha^2=\beta^2+\gamma^2-2\beta\gamma\cdot\sigma\upsilon\nu 120^0=\beta^2+\gamma^2+2\beta\gamma\cdot\frac{1}{2}=\beta^2+\gamma^2+\beta\gamma.$

Ἐὰν $\beta\neq\gamma$, τότε $\gamma\ \beta\neq 0$ καὶ ἄρα

$$\alpha^2(\gamma-\beta)=(\gamma^2+\beta^2+\beta\gamma)(\gamma-\beta)=\gamma^3-\beta^3$$

ἢ $\alpha^2\gamma-\alpha^2\beta=\gamma^3-\beta^3$ ἢ $\alpha^2\gamma-\gamma^3=\alpha^2\beta-\beta^3$

ἢ $\gamma(\alpha^2-\gamma^2)=\beta(\alpha^2-\beta^2).$

130. Ἐὰν αἱ πλευραὶ τριγώνου ἀποτελοῦν ἀριθμητικὴν πρόοδον, νὰ ἀποδειχθῇ ὅτι τὰ ἡμίτονα τῶν γωνιῶν τῶν ἀπέναντι τῶν πλευρῶν τούτων ἀποτελοῦν ἀριθμητικὴν πρόοδον :

Λύσις. Ἔστω ὅτι $\beta+\gamma=2\alpha$, τότε θὰ εἶναι :

$$2R\eta\mu B+2R\eta\mu\Gamma=2\ 2R\eta\mu A \quad \text{ἢ} \quad \eta\mu B+\eta\mu\Gamma=2\eta\mu A.$$

Ἄρα τὰ ημΒ, ημΒ, ημΓ ἀποτελοῦν ἀριθμητικὴν περίοδον.

131. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $\alpha^2+\gamma^2=2\beta^2$, νὰ ἀποδειχθῇ ὅτι :

$$\sigma\varphi A+\sigma\varphi\Gamma=2\sigma\varphi B.$$

Λύσις. Ἐκ τῆς σχέσεως $\alpha^2+\gamma^2=2\beta^2$, ἕπεναι ὅτι :

$$\alpha^2-\beta^2=\beta^2-\gamma^2 \quad \text{ἢ} \quad 4R^2\eta\mu^2 A-4R^2\eta\mu^2 B=4R^2\eta\mu^2 B-4R^2\eta\mu^2\Gamma$$

ἢ $\eta\mu^2 A-\eta\mu^2 B=\eta\mu^2 B-\eta\mu^2\Gamma$

ἢ $\eta\mu(A+B)\eta\mu(A-B)=\eta\mu(B+\Gamma)\eta\mu(B-\Gamma)$

ἢ $$\frac{\eta\mu(A-B)}{\eta\mu A\eta\mu B}=\frac{\eta\mu(B-\Gamma)}{\eta\mu B\eta\mu\Gamma} \qquad (1)$$

καθόσον εἶναι $\eta\mu(A+B)=\eta\mu\Gamma$ καὶ $\eta\mu(B+\Gamma)=\eta\mu A$,

Ἡ (1) γράφεται :

$$\frac{\eta\mu A\sigma\upsilon\nu B-\eta\mu B\sigma\upsilon\nu A}{\eta\mu A\eta\mu B}=\frac{\eta\mu B\sigma\upsilon\nu\Gamma-\eta\mu\Gamma\sigma\upsilon\nu B}{\eta\mu B\eta\mu\Gamma}$$

ἢ $\sigma\varphi B-\sigma\varphi A=\sigma\varphi\Gamma-\sigma\varphi B \Longleftarrow \sigma\varphi A+\sigma\varphi\Gamma=2\sigma\varphi B.$

Πῶς θὰ ἀποδείξητε τὸ ἀντίστροφον :

132. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $\alpha+\gamma=2\beta$, νὰ ἀποδειχθῇ ὅτι:

1. $$\text{συν}A\,\text{σφ}\frac{A}{2}+\text{συν}\Gamma\,\text{σφ}\frac{\Gamma}{2}=2\,\text{συν}B\,\text{σφ}\frac{B}{2}.$$

Λύσις. Ἡ δοθεῖσα σχέσις $\alpha+\gamma=2\beta$ γράφεται:

$$-\alpha-\gamma=-2\beta \quad \text{ἢ} \quad (\tau-\alpha)+(\tau-\gamma)=2(\tau-\beta)$$

ἢ $$\frac{\tau(\tau-\alpha)}{E}+\frac{\tau(\tau-\gamma)}{E}=2\cdot\frac{\tau(\tau-\beta)}{E}$$

ἢ $$\sqrt{\frac{\tau(\tau-\alpha)}{(\tau-\beta)(\tau-\gamma)}}+\sqrt{\frac{\tau(\tau-\gamma)}{(\tau-\alpha)(\tau-\beta)}}=2\sqrt{\frac{\tau(\tau-\beta)}{(\tau-\alpha)(\tau-\gamma)}}$$

ἢ $$\text{σφ}\frac{A}{2}+\text{σφ}\frac{\Gamma}{2}=2\,\text{σφ}\frac{B}{2} \qquad (1)$$

Ἐκ τῆς $\alpha+\gamma=2\beta \Longrightarrow \text{ημ}A+\text{ημ}\Gamma=2\,\text{ημ}B$

ἢ $$-\text{ημ}A-\text{ημ}\Gamma=-2\,\text{ημ}B. \qquad (2)$$

Διὰ προσθέσεως κατὰ μέλη τῶν (1) καὶ (2), λαμβάνομεν:

$$\left(\text{σφ}\frac{A}{2}-\text{ημ}A\right)+\left(\text{σφ}\frac{\Gamma}{2}-\text{ημ}\Gamma\right)=2\left(\text{σφ}\frac{B}{2}-\text{ημ}B\right)$$

ἢ $$\left(\frac{\text{συν}\frac{A}{2}}{\text{ημ}\frac{A}{2}}-2\,\text{ημ}\frac{A}{2}\,\text{συν}\frac{A}{2}\right)+\left(\frac{\text{συν}\frac{\Gamma}{2}}{\text{ημ}\frac{\Gamma}{2}}-2\,\text{ημ}\frac{\Gamma}{2}\,\text{συν}\frac{\Gamma}{2}\right)=$$

$$=2\left(\frac{\text{συν}\frac{B}{2}}{\text{ημ}\frac{\Gamma}{2}}-2\,\text{ημ}\frac{B}{2}\,\text{συν}\frac{B}{2}\right)$$

ἢ $$\left(1-2\,\text{ημ}^2\frac{A}{2}\right)\cdot\text{σφ}\frac{A}{2}+\left(1-2\,\text{ημ}^2\frac{\Gamma}{2}\right)\text{σφ}\frac{\Gamma}{2}=2\left(1-2\,\text{ημ}^2\frac{B}{2}\right)\text{σφ}\frac{B}{2}$$

ἢ $$\text{συν}A\cdot\text{σφ}\frac{A}{2}+\text{συν}\Gamma\,\text{σφ}\frac{\Gamma}{2}=2\,\text{συν}B\,\text{σφ}\frac{B}{2}.$$

Πῶς θὰ ἀποδείξητε τὸ ἀντίστροφον;

2. $$\alpha\,\text{συν}^2\frac{\Gamma}{2}+\gamma\,\text{συν}^2\frac{A}{2}=\frac{3\beta}{2}.$$

Λύσις. Γνωρίζομεν ὅτι: $\alpha+\gamma=2\beta$

ἢ $$\alpha+\beta+\gamma=2\beta+\beta=3\beta \quad \text{ἢ} \quad 2\tau=3\beta \quad \text{ἢ} \quad \tau=\frac{3\beta}{2} \quad \text{ἢ} \quad \tau\beta=\frac{3\beta^2}{2}$$

ἢ $$\tau(2\tau-\alpha-\gamma)=\frac{3\beta^2}{2} \quad \text{ἢ} \quad \tau(\tau-\gamma)+\tau(\tau-\alpha)=\frac{3\beta^2}{2}$$

ἢ $$\frac{\tau(\tau-\gamma)}{\beta}+\frac{\tau(\tau-\alpha)}{\beta}=\frac{3\beta}{2} \quad \text{ἢ} \quad \alpha\cdot\frac{\tau(\tau-\gamma)}{\alpha\beta}+\gamma\cdot\frac{\tau(\tau-\alpha)}{\beta\gamma}=\frac{3\beta}{2}$$

ἢ $$\alpha\cdot\text{συν}^2\frac{\Gamma}{2}+\gamma\,\text{συν}^2\frac{A}{2}=\frac{3\beta}{2}.$$

Πῶς θὰ ἀποδείξητε τὸ ἀντίστροφον;

3. $$\sigma\varphi\frac{A}{2}+\sigma\varphi\frac{\Gamma}{2}=2\sigma\varphi\frac{B}{2}.$$

Λύσις. Ἐκ τῆς δοθείσης σχέσεως $\alpha+\gamma=2\beta$, ἔχομεν:

$$\alpha-\beta=\beta-\gamma \quad \text{ἢ} \quad \eta\mu A-\eta\mu B=\eta\mu B-\eta\mu\Gamma$$

ἢ $$2\eta\mu\frac{A-B}{2}\,\sigma\upsilon\nu\frac{A+B}{2}=2\eta\mu\frac{B-\Gamma}{2}\,\sigma\upsilon\nu\frac{B+\Gamma}{2}$$

ἢ $$\eta\mu\left(\frac{A}{2}-\frac{B}{2}\right)\eta\mu\frac{\Gamma}{2}=\eta\mu\left(\frac{B}{2}-\frac{\Gamma}{2}\right)\eta\mu\frac{A}{2}$$

ἢ $$\eta\mu\frac{A}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\eta\mu\frac{\Gamma}{2}-\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{A}{2}\,\eta\mu\frac{\Gamma}{2}=$$

$$=\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}\,\eta\mu\frac{A}{2}-\eta\mu\frac{\Gamma}{2}\,\sigma\upsilon\nu\frac{B}{2}\,\eta\mu\frac{A}{2},$$

ἤ, διαιροῦντες ἀμφότερα τὰ μέλη διὰ $\eta\mu\frac{A}{2}\,\eta\mu\frac{B}{2}\,\eta\mu\frac{\Gamma}{2}\neq 0,$

$$\sigma\varphi\frac{B}{2}-\sigma\varphi\frac{A}{2}=\sigma\varphi\frac{\Gamma}{2}-\sigma\varphi\frac{B}{2}$$

ἐξ οὗ: $$\sigma\varphi\frac{A}{2}+\sigma\varphi\frac{\Gamma}{2}=2\sigma\varphi\frac{B}{2}.$$

Πῶς θὰ ἀποδείξητε τὸ ἀντίστροφον;

4. $$\varepsilon\varphi\frac{A}{2}\cdot\varepsilon\varphi\frac{\Gamma}{2}=\frac{1}{3}.$$

Λύσις. Γνωρίζομεν ὅτι: $\alpha+\gamma=2\beta$ ἢ $\eta\mu A+\eta\mu\Gamma=2\eta\mu B$

ἢ $$2\eta\mu\frac{A+\Gamma}{2}\,\sigma\upsilon\nu\frac{A-\Gamma}{2}=2\cdot 2\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{B}{2}$$

ἢ $$\sigma\upsilon\nu\frac{B}{2}\,\sigma\upsilon\nu\frac{A-\Gamma}{2}=2\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{B}{2} \quad \text{ἢ} \quad \sigma\upsilon\nu\frac{A-\Gamma}{2}=2\eta\mu\frac{B}{2}$$

ἢ $$\sigma\upsilon\nu\left(\frac{A}{2}-\frac{\Gamma}{2}\right)=2\sigma\upsilon\nu\left(\frac{A}{2}+\frac{\Gamma}{2}\right)$$

ἢ $$\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}+\eta\mu\frac{A}{2}\,\eta\mu\frac{\Gamma}{2}=2\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}-2\eta\mu\frac{A}{2}\,\eta\mu\frac{\Gamma}{2}$$

ἐξ οὗ: $$3\eta\mu\frac{A}{2}\,\eta\mu\frac{\Gamma}{2}=\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}.$$

Διαιροῦντες ἀμφότερα τὰ μέλη ταύτης διὰ $3\sigma\upsilon\nu\frac{A}{2}\,\sigma\upsilon\nu\frac{\Gamma}{2}$, λαμβάνομεν: $$\varepsilon\varphi\frac{A}{2}\,\varepsilon\varphi\frac{\Gamma}{2}=\frac{1}{3}.$$

Ἀποδείξατε τὸ ἀντίστροφον.

133. Ἐὰν αἱ πλευραὶ α, β, γ τριγώνου ΑΒΓ ἀποτελοῦν ἁρμονικὴν πρόοδον, νὰ ἀποδειχθῇ ὅτι καὶ οἱ ἀριθμοὶ

$$\eta\mu^2 \frac{A}{2},\ \eta\mu^2 \frac{B}{2},\ \eta\mu^2 \frac{\Gamma}{2}$$

ἀποτελοῦν ἁρμονικὴν πρόοδον.

Λύσις. Ἐπειδὴ οἱ α, β, γ ἀποτελοῦν ἁρμονικὴν πρόοδον, οἱ ἀντίστροφοι αὐτῶν θὰ ἀποτελοῦν ἀριθμητικὴν πρόοδον.

Δηλαδή: $\frac{1}{\gamma}+\frac{1}{\alpha}=\frac{2}{\beta}$ ἢ $\frac{\tau}{\gamma}+\frac{\tau}{\alpha}=2\cdot\frac{\tau}{\beta}$

ἢ $\left(\frac{\tau}{\gamma}-1\right)+\left(\frac{\tau}{\alpha}-1\right)=2\left(\frac{\tau}{\beta}\right)-2$ ἢ $\frac{\tau-\alpha}{\alpha}+\frac{\tau-\gamma}{\gamma}=2\,\frac{\tau-\beta}{\beta}$ (1)

Διαιροῦντες ἀμφότερα τὰ μέλη ταύτης διὰ $\frac{(\tau-\alpha)(\tau-\beta)(\tau-\gamma)}{\alpha\beta\gamma}$,

λαμβάνομεν: $\frac{\beta\gamma}{(\tau-\beta)(\tau-\gamma)}+\frac{\alpha\beta}{(\tau-\alpha)(\tau-\beta)}=2\cdot\frac{\gamma\alpha}{(\tau-\gamma)(\tau-\alpha)}$

ἢ
$$\frac{1}{\eta\mu^2\frac{A}{2}}+\frac{1}{\eta\mu^2\frac{\Gamma}{2}}=2\cdot\frac{1}{\eta\mu^2\frac{B}{2}} \qquad (2)$$

Ἡ σχέσις αὕτη φανερώνει ὅτι οἱ ἀριθμοὶ

$$\eta\mu^2 \frac{A}{2},\ \eta\mu^2 \frac{B}{2},\ \eta\mu^2 \frac{\Gamma}{2}$$

ἀποτελοῦν ἁρμονικὴν πρόοδον.

134. Ἐὰν εἰς τρίγωνον ΑΒΓ εἶναι $\alpha+\gamma=2\beta$ καὶ $A-\Gamma=90^\circ$, νὰ ἀποδειχθῇ ὅτι:

$$\frac{\alpha}{\sqrt{7}+1}=\frac{\beta}{7}=\frac{\gamma}{\sqrt{7}-1}.$$

Λύσις. Ἐκ τῆς $\alpha+\gamma=2\beta \Longrightarrow \eta\mu A+\eta\mu\Gamma=2\eta\mu B=2\eta\mu(A+\Gamma)$

ἢ
$$2\eta\mu\frac{A+\Gamma}{2}\,\sigma\upsilon\nu\frac{A-\Gamma}{2}=4\eta\mu\frac{A+\Gamma}{2}\,\sigma\upsilon\nu\frac{A+\Gamma}{2}$$

ἢ
$$\sigma\upsilon\nu\frac{A-\Gamma}{2}=2\sigma\upsilon\nu\frac{A+\Gamma}{2}=2\eta\mu\frac{B}{2}$$

ἢ
$$\sigma\upsilon\nu 45^\circ=2\eta\mu\frac{B}{2} \quad \text{ἢ} \quad \frac{\sqrt{2}}{2}=2\eta\mu\frac{B}{2} \quad \text{ἢ} \quad \eta\mu\frac{B}{2}=\frac{\sqrt{2}}{4}$$

καὶ κατ' ἀκολουθίαν:

$$\sigma\upsilon\nu\frac{B}{2}=\sqrt{1-\eta\mu^2\frac{B}{2}}=\sqrt{1-\frac{2}{16}}=\sqrt{\frac{14}{16}}=\frac{\sqrt{14}}{4}$$

καὶ
$$\eta\mu A+\eta\mu\Gamma=2\eta\mu B=4\eta\mu\frac{B}{2}\,\sigma\upsilon\nu\frac{B}{2}=4\cdot\frac{\sqrt{2}}{4}\cdot\frac{\sqrt{14}}{4}=$$

$$=\frac{\sqrt{28}}{4}=\frac{2\sqrt{7}}{4}=\frac{\sqrt{7}}{2}.$$

'Αλλ' $\eta\mu A - \eta\mu\Gamma = 2\eta\mu\frac{A-\Gamma}{2}\sigma\upsilon\nu\frac{A+\Gamma}{2} = 2\cdot\eta\mu 45^0\eta\mu\frac{B}{2} =$

$$= 2\cdot\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{4} = \frac{1}{2}.$$

'Επειδὴ $\eta\mu A + \eta\mu\Gamma = \frac{\sqrt{7}}{2}$ καὶ $\eta\mu A - \eta\mu\Gamma = \frac{1}{2}$,

$$\eta\mu A = \frac{\sqrt{7}+1}{4} \quad \text{καὶ} \quad \eta\mu\Gamma = \frac{\sqrt{7}-1}{4}.$$

'Αλλ' εἶναι: $\frac{\alpha}{\eta\mu A} = \frac{\beta}{\eta\mu B} = \frac{\gamma}{\eta\mu\Gamma}$

ἢ $\frac{\alpha}{\frac{\sqrt{7}+1}{4}} = \frac{\beta}{\frac{\sqrt{7}}{4}} = \frac{\gamma}{\frac{\sqrt{7}-1}{4}}$ ἢ $\frac{\alpha}{\sqrt{7}+1} = \frac{\beta}{\sqrt{7}} = \frac{\gamma}{\sqrt{7}-1}$.

135. 'Εὰν εἰς τρίγωνον ΑΒΓ εἶναι $\Gamma = 60^0$, νὰ ἀποδειχθῇ ὅτι:

$$\frac{1}{\alpha+\gamma} + \frac{1}{\beta+\gamma} = \frac{3}{\alpha+\beta+\gamma}$$

καὶ ἀντιστρόφως.

Λύσις. Γνωρίζομεν ὅτι: $\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\sigma\upsilon\nu\Gamma$

ἢ $\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\cdot\frac{1}{2}$ ἢ $\gamma^2 = \alpha^2 + \beta^2 - \alpha\beta$ ἢ $\alpha^2 + \beta^2 = \gamma^2 + \alpha\beta$.

Προσθέτομεν εἰς ἀμφότερα τὰ μέλη ταύτης τὴν παράστασιν:

$$2\gamma^2 + 3\alpha\gamma + 3\beta\gamma + 2\alpha\beta$$

καὶ λαμβάνομεν:

$$2\gamma^2 + 3\alpha\gamma + 3\beta\gamma + 2\alpha\beta + \alpha^2 + \beta^2 = \gamma^2 + \alpha\beta + 2\gamma^2 + 3\alpha\gamma + 3\beta\gamma + 2\alpha\beta$$

ἢ $2\gamma^2 + \alpha\gamma + \beta\gamma + 2\alpha\gamma + 2\beta\gamma + \alpha^2 + \beta^2 + 2\alpha\beta = 3\gamma^2 + 3\alpha\gamma + 3\beta\gamma + 3\alpha\beta$

ἢ $2\gamma^2 + (\alpha+\beta)\gamma + 2\gamma(\alpha+\beta) + (\alpha+\beta)^2 = 3(\gamma^2 + \alpha\gamma + \beta\gamma + \alpha\beta)$

ἢ $(2\gamma+\alpha+\beta)(\gamma+\alpha+\beta) = 3(\alpha+\gamma)(\beta+\gamma)$

ἢ $\frac{2\gamma+\alpha+\beta}{(\alpha+\gamma)(\beta+\gamma)} = \frac{3}{\alpha+\beta+\gamma}$

ἢ $\frac{(\beta+\gamma)+(\alpha+\gamma)}{(\alpha+\gamma)(\beta+\gamma)} = \frac{3}{\alpha+\beta+\gamma}$

ἢ $\frac{1}{\alpha+\gamma} + \frac{1}{\beta+\gamma} = \frac{3}{\alpha+\beta+\gamma}$.

'Αντιστρόφως: 'Εκτελοῦντες τὰς πράξεις κατ' ἀντίστροφον τάξιν, λαμβάνομεν: $\gamma^2 = \alpha^2 + \beta^2 - \alpha\beta$.

'Αλλὰ $\gamma^2 = \alpha^2 + \beta^2 - 2\alpha\beta\sigma\upsilon\nu\Gamma$ ἢ $\alpha^2 + \beta^2 - \alpha\beta = \alpha^2 + \beta^2 - 2\alpha\beta\sigma\upsilon\nu\Gamma$

ἢ $\sigma\upsilon\nu\Gamma = \frac{1}{2} \Rightarrow \Gamma = 60^0$.

136. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι :

1. $\alpha^3\text{συν}(B-\Gamma)+\beta^3\text{συν}(\Gamma-A)+\gamma^3\text{συν}(A-B)=3\alpha\beta\gamma.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\alpha^3\text{συν}(B-\Gamma)=8R^3\text{ημ}^3A\text{συν}(B-\Gamma)=8R^3\text{ημ}^2A\cdot\text{ημ}A\text{συν}(B-\Gamma)=$$
$$=8R^3\text{ημ}^2A\text{ημ}(B+\Gamma)\text{συν}(B-\Gamma)$$
$$=4R^3\text{ημ}^2A\cdot 2\text{ημ}(B+\Gamma)\text{συν}(B-\Gamma)$$
$$=4R^3\text{ημ}^2A(\text{ημ}2B+\text{ημ}2\Gamma)$$
$$=4R^3\text{ημ}^2A\cdot(2\text{ημ}B\text{συν}B+2\text{ημ}\Gamma\text{συν}\Gamma)$$
$$=8R^3\text{ημ}^2A\text{ημ}B\text{συν}B+8R^3\text{ημ}^2A\text{ημ}\Gamma\text{συν}\Gamma.$$

Καὶ διὰ κυκλικῆς ἐναλλαγῆς τῶν γραμμάτων Α,Β,Γ ἔχομεν :

$$\left.\begin{array}{l}\Sigma\alpha^3\text{συν}(B-\Gamma)=8R^3\text{ημ}^2A\text{ημ}B\text{συν}B+8R^3\text{ημ}^2A\text{ημ}\Gamma\text{συν}\Gamma+\\ \quad +8R^3\text{ημ}^2B\text{ημ}\Gamma\text{συν}\Gamma+8R^3\text{ημ}^2B\text{ημ}A\text{συν}A+\\ \quad +8R^3\text{ημ}^2\Gamma\text{ημ}A\text{συν}A+8R^3\text{ημ}^2\Gamma\text{ημ}B\text{συν}B\end{array}\right\}=$$

$$=8R^3\left\{\begin{array}{l}\text{ημ}A\text{ημ}B(\text{ημ}A\text{συν}B+\text{ημ}B\text{συν}A)+\\ +\text{ημ}B\text{ημ}\Gamma(\text{ημ}B\text{συν}\Gamma+\text{ημ}\Gamma\text{συν}B)+\\ +\text{ημ}\Gamma\text{ημ}A(\text{ημ}\Gamma\text{συν}A+\text{ημ}A\text{συν}\Gamma)\end{array}\right\}=$$

$$=8R^3[\text{ημ}A\text{ημ}B\cdot\text{ημ}(A+B)+\text{ημ}B\text{ημ}\Gamma\cdot\text{ημ}(B+\Gamma)+\text{ημ}\Gamma\text{ημ}A\text{ημ}(\Gamma+A)]$$
$$=8R^3\cdot(\text{ημ}A\text{ημ}B\text{ημ}\Gamma+\text{ημ}B\text{ημ}\Gamma\text{ημ}A+\text{ημ}\Gamma\text{ημ}A\text{ημ}B)$$
$$=24R^3\text{ημ}A\text{ημ}B\text{ημ}\Gamma=3\cdot 2R\text{ημ}A\cdot 2R\text{ημ}B\cdot 2R\text{ημ}\Gamma$$
$$=3\cdot\alpha\cdot\beta\cdot\gamma=3\alpha\beta\gamma.$$

2. $\beta^2\text{συν}2B+\gamma^2\text{συν}2\Gamma+2\beta\gamma\text{συν}(B-\Gamma)=\alpha^2\text{συν}2(B-\Gamma).$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\beta^2\text{συν}2B+\gamma^2\text{συν}2\Gamma+2\beta\gamma\text{συν}(B-\Gamma)=$$
$$=4R^2\text{ημ}^2B\text{συν}2B+4R^2\text{ημ}^2\Gamma\text{συν}2\Gamma+8R^2\text{ημ}B\text{ημ}\Gamma\text{συν}(B-\Gamma)=$$
$$=2R^2[2\text{ημ}^2B\text{συν}2B+2\text{ημ}^2\Gamma\text{συν}2\Gamma+4\text{ημ}B\text{ημ}\Gamma(\text{συν}B\text{συν}\Gamma+\text{ημ}B\text{ημ}\Gamma)]=$$
$$=2R^2[(1-\text{συν}2B)\text{συν}2B+(1-\text{συν}2\Gamma)\text{συν}2\Gamma+2\text{ημ}B\text{συν}B\cdot 2\text{ημ}2\Gamma+2\text{ημ}^2B\cdot 2\text{ημ}^2\Gamma]$$
$$=2R^2[(1-\text{συν}2B)\text{συν}2B+(1-\text{συν}2\Gamma)\text{συν}2\Gamma+\text{ημ}2B\text{ημ}2\Gamma+(1-\text{συν}2B)(1-\text{συν}2\Gamma)]$$
$$=2R^2[\text{συν}2B-\text{συν}^2 2B+\text{συν}2\Gamma-\text{συν}^2 2\Gamma+\text{ημ}2B\text{ημ}2\Gamma+$$
$$+1-\text{συν}2B-\text{συν}2\Gamma+\text{συν}2B\text{συν}2\Gamma]$$
$$=2R^2[(\text{συν}2B\text{συν}2\Gamma+\text{ημ}2B\text{ημ}2\Gamma)+1-(\text{συν}^2 2B+\text{συν}^2 2\Gamma)]$$
$$=2R^2\left[\text{συν}2(B-\Gamma)+1-\left(\frac{1+\text{συν}4B}{2}+\frac{1+\text{συν}4\Gamma}{2}\right)\right]$$
$$=2R^2[\text{συν}2(B-\Gamma)+1-1-\frac{1}{2}(\text{συν}4B+\text{συν}4\Gamma)]$$
$$=2R^2[\text{συν}2(B-\Gamma)-\text{συν}2(B+\Gamma)\text{συν}2(B-\Gamma)]=2R^2\text{συν}2(B-\Gamma)[1-\text{συν}2(B+\Gamma)]$$
$$=2R^2\text{συν}2(B-\Gamma)\cdot 2\text{ημ}^2(B+\Gamma)=2R^2\cdot\text{συν}2(B-\Gamma)\cdot 2\text{ημ}^2A=4R^2\text{ημ}^2A\text{συν}2(B-\Gamma)$$
$$=\alpha^2\text{συν}2(B-\Gamma).$$

3. $\Sigma\, \alpha^2 \sigma\upsilon\nu^2 A + 2\Sigma\, \beta\gamma\sigma\upsilon\nu 2A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma = 0.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\alpha^2\sigma\upsilon\nu^2A+\beta^2\sigma\upsilon\nu^2B+\gamma^2\sigma\upsilon\nu^2\Gamma+2\beta\gamma\sigma\upsilon\nu2A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma+2\gamma\alpha\sigma\upsilon\nu2B\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A+$$
$$+2\alpha\beta\sigma\upsilon\nu2\Gamma\sigma\upsilon\nu A\sigma\upsilon\nu B=$$
$$=(\alpha\sigma\upsilon\nu A+\beta\sigma\upsilon\nu B+\gamma\sigma\upsilon\nu\Gamma)^2-2\alpha\beta\sigma\upsilon\nu A\sigma\upsilon\nu B-2\alpha\gamma\sigma\upsilon\nu A\sigma\upsilon\nu\Gamma-2\beta\gamma\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma+$$
$$+2\beta\gamma\sigma\upsilon\nu2A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma+2\gamma\alpha\sigma\upsilon\nu2B\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A+2\alpha\beta\sigma\upsilon\nu2\Gamma\sigma\upsilon\nu A\sigma\upsilon\nu B=$$
$$=R^2(\eta\mu2A+\eta\mu2B+\eta\mu2\Gamma)^2+2\alpha\beta\sigma\upsilon\nu A\sigma\upsilon\nu B(\sigma\upsilon\nu2\Gamma-1)+$$
$$+2\beta\gamma\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma(\sigma\upsilon\nu2A-1)+2\gamma\alpha\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A(\sigma\upsilon\nu2B-1)=$$
$$=16R^2\eta\mu^2A\eta\mu^2B\eta\mu^2\Gamma-2\alpha\beta\sigma\upsilon\nu A\sigma\upsilon\nu B\cdot2\eta\mu^2\Gamma-2\beta\gamma\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\cdot2\eta\mu^2A-$$
$$-2\gamma\alpha\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A\cdot2\eta\mu^2B=$$
$$=16R^2\eta\mu^2A\eta\mu^2B\eta\mu^2\Gamma-4\alpha\beta\sigma\upsilon\nu A\sigma\upsilon\nu B\eta\mu^2\Gamma-4\beta\gamma\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\eta\mu^2A-$$
$$-4\gamma\alpha\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A\eta\mu^2B$$
$$=16R^2\eta\mu^2A\eta\mu^2B\eta\mu^2\Gamma-16R^2\eta\mu A\eta\mu B\sigma\upsilon\nu A\sigma\upsilon\nu B\eta\mu^2\Gamma-$$
$$-16R^2\eta\mu B\eta\mu\Gamma\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\eta\mu^2A-16R^2\eta\mu\Gamma\eta\mu A\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A\eta\mu^2B$$
$$=-16R^2\eta\mu A\eta\mu B\eta\mu\Gamma(\sigma\upsilon\nu A\sigma\upsilon\nu B\eta\mu\Gamma-\eta\mu A\eta\mu B\eta\mu\Gamma+\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\eta\mu A+$$
$$+\sigma\upsilon\nu\Gamma\sigma\upsilon\nu A\eta\mu B)$$
$$=-16R^2\eta\mu A\eta\mu B\eta\mu\Gamma[\eta\mu\Gamma(\sigma\upsilon\nu A\sigma\upsilon\nu B-\eta\mu A\eta\mu B)+\sigma\upsilon\nu\Gamma(\eta\mu A\sigma\upsilon\nu B+\eta\mu B\,\sigma\upsilon\nu A)]$$
$$=-16R^2\eta\mu A\eta\mu B\eta\mu\Gamma[\eta\mu\Gamma\sigma\upsilon\nu(A+B)+\sigma\upsilon\nu\Gamma\eta\mu(A+B)]=$$
$$=-16R^3\eta\mu A\eta\mu B\eta\mu\Gamma(-\eta\mu\Gamma\sigma\upsilon\nu\Gamma+\sigma\upsilon\nu\Gamma\eta\mu\Gamma)=0.$$

4. $\alpha^6+\beta^6+\gamma^6-2\Sigma\,\beta^3\gamma^3\sigma\upsilon\nu A=\alpha^2\beta^2\gamma^2(1-8\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma).$

Λύσις. Ἔχομεν διαδοχικῶς :

$$\alpha^2\beta^2\gamma^2(1-8\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma)=\alpha^2\beta^2\gamma^2-8\alpha^2\beta^2\gamma^2\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma=$$
$$=\alpha^2\beta^2\gamma^2-8\alpha^2\beta^2\gamma^2\cdot\frac{\beta^2+\gamma^2-\alpha^2}{2\beta\gamma}\cdot\frac{\gamma^2+\alpha^2-\beta^2}{2\gamma\alpha}\cdot\frac{\alpha^2+\beta^2-\gamma^2}{2\alpha\beta}$$
$$=\alpha^2\beta^2\gamma^2-(\beta^2+\gamma^2-\alpha^2)(\gamma^2+\alpha^2-\beta^2)(\alpha^2+\beta^2-\gamma^2)$$
$$=\alpha^6+\beta^6+\gamma^6-\beta^2\gamma^4-\beta^4\gamma^2+\alpha^2\beta^2\gamma^2-\alpha^2\gamma^4-\alpha^4\gamma^2+\alpha^2\beta^2\gamma^2-\alpha^2\beta^4-\alpha^4\beta^2+\alpha^2\beta^2\gamma^2$$
$$=\alpha^6+\beta^6+\gamma^6-\beta^2\gamma^2(\beta^2+\gamma^2-\alpha^2)-\gamma^2\alpha^2(\gamma^2+\alpha^2-\beta^2)-\alpha^2\beta^2(\alpha^2+\beta^2-\gamma^2)$$
$$=\alpha^6+\beta^6+\gamma^6-\beta^2\gamma^2\cdot2\beta\gamma\sigma\upsilon\nu A-\gamma^2\alpha^2\cdot2\gamma\alpha\sigma\upsilon\nu B-\alpha^2\beta^2\cdot2\alpha\beta\sigma\upsilon\nu\Gamma$$
$$=\alpha^6+\beta^6+\gamma^6-2\beta^3\gamma^3\sigma\upsilon\nu A-2\gamma^3\alpha^3\sigma\upsilon\nu B-2\alpha^3\beta^3\sigma\upsilon\nu\Gamma$$
$$=\alpha^6+\beta^6+\gamma^6-2\Sigma\beta^3\gamma^3\sigma\upsilon\nu A.$$

5. $\Sigma\,\eta\mu^4A+4\Pi\eta\mu^2A=2\Sigma\,\eta\mu^2B\eta\mu^2\Gamma.$

Λύσις. Ἐὰν α, β, γ εἶναι αἱ πλευραὶ τοῦ τριγώνου ΑΒΓ, τότε, βάσει τῆς ταυτότητος τοῦ de Moivre, θὰ ἔχωμεν :

$$2\beta^2\gamma^2+2\gamma^2\alpha^2+2\alpha^2\beta^2-\alpha^4-\beta^4-\gamma^4=(\alpha+\beta+\gamma)(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)(\alpha+\beta-\gamma)$$

ἢ $32R^4 ημ^2Bημ^2Γ + 32R^4ημ^2Γημ^2A + 32ημ^2Aημ^2B - 16R^4ημ^4A - 16R^4ημ^4B -$
$- 16R^4ημ^4Γ =$

$= 16R^4(ημA + ημB + ημΓ)(ημB + ημΓ - ημA)(ημΓ + ημA - ημB)(ημA + ημB - ημΓ)$

ἢ $2ημ^2Bημ^2Γ + 2ημ^2Γημ^2A + 2ημ^2Aημ^2B - ημ^4A - ημ^4B - ημ^4Γ =$

$$= 4συν\frac{A}{2}συν\frac{B}{2}συν\frac{Γ}{2} \cdot 4ημ\frac{B}{2}ημ\frac{Γ}{2}συν\frac{A}{2} \cdot$$

$$\cdot 4ημ\frac{Γ}{2}ημ\frac{A}{2}συν\frac{B}{2} \cdot 4ημ\frac{A}{2}ημ\frac{B}{2}συν\frac{Γ}{2} =$$

$$= 4\ 4ημ^2\frac{A}{2}συν^2\frac{A}{2} \cdot 4ημ^2\frac{B}{2}συν^2\frac{B}{2} \cdot 4ημ^2\frac{Γ}{2}συν^2\frac{Γ}{2} = 4ημ^2Aημ^2Bημ^2Γ.$$

Κατ' ἀκολουθίαν :

$$ημ^4A + ημ^4B + ημ^4Γ + 4ημ^2Aημ^2Bημ^2Γ = 2ημ^2Bημ^2Γ + 2ημ^2Γημ^2A + 2ημ^2Aημ^2B$$

ἢ $$Σ\,ημ^4A + 4Πημ^2A = 2\,Σ\,ημ^2Bημ^2Γ.$$

137. Ἐὰν συνA=συναημβ, συνB=συνβημγ, συνΓ=συνγημα, καὶ A+B+Γ=π, νὰ ἀποδειχθῇ ὅτι : εφαεφβεφγ=1.

Λύσις. Ἐκ τῆς σχέσεως $A + B + Γ = π$, ἔχομεν :

$A + B = π - Γ$ ἢ $συν(A+B) = συν(π - Γ) = -συνΓ$ ἢ $συν(A+B) + συνΓ = 0$

ἢ $συνAσυνB - ημAημB + συνΓ = 0$ ἢ $συνAσυνB + συνΓ = ημAημB$

ἢ $(συνAσυνB + συνΓ)^2 = ημ^2Aημ^2B = (1 - συν^2A)(1 - συν^2B)$

ἢ $συν^2A + συν^2B + συν^2Γ + 2συνAσυνBσυνΓ = 1$

ἢ $συν^2αημ^2β + συν^2βημ^2γ + συν^2γημ^2α + 2συνασυνβσυνγημαημβημγ = 1.$

Ἐὰν $συν^2ασυν^2βσυν^2γ \neq 0$, διαιροῦντες ἀμφότερα τὰ μέλη ταύτης διὰ $συν^2ασυν^2βσυν^2γ$, λαμβάνομεν :

$$\frac{εφ^2β}{συν^2γ} + \frac{εφ^2γ}{συν^2α} + \frac{εφ^2α}{συν^2β} + 2εφαεφβεφγ = \frac{1}{συν^2ασυν^2βσυν^2γ}$$

ἢ $(1 + εφ^2γ)εφ^2β + (1 + εφ^2α)εφ^2γ + (1 + εφ^2β)εφ^2α + 2εφαεφβεφγ =$

$= (1 + εφ^2α)(1 + εφ^2β)(1 + εφ^2γ)$ ἢ $2εφαεφβεφγ = 1 + εφ^2αεφ^2βεφ^2γ$

ἢ $(1 - εφαεφβεφγ)^2 = 0 \implies εφαεφβεφγ = 1.$

138. Ἐὰν συνA=εφβεφγ, συνB=εφγεφα, καὶ συνΓ=εφαεφβ καὶ A+B+Γ=π, νὰ ἀποδειχθῇ ὅτι : ημ²α+ημ²β+ημ²γ=1.

Λύσις. Ἐκ τῆς $A + B + Γ = π \implies A + B = π - Γ$ ἢ

$συν(A+B) = συν(π - Γ) = -συνΓ$ ἢ $συνAσυνB + συνΓ = ημAημB$

ἢ, ὅπως εἰς τὴν προηγουμένην ἄσκησιν,

$$συν^2A + συν^2B + συν^2Γ + 2συνAσυνBσυνΓ = 1,$$

ἢ $$\frac{ημ^2βημ^2γ}{συν^2βσυν^2γ} + \frac{ημ^2γημ^2α}{συν^2γσυν^2α} + \frac{ημ^2αημ^2β}{συν^2ασυν^2β} + \frac{2ημ^2αημ^2βημ^2γ}{συν^2ασυν^2βσυν^2γ} = 1$$

ἢ $ημ^2βημ^2γσυν^2α + ημ^2γημ^2ασυν^2β + ημ^2αημ^2βσυν^2γ +$
$+ 2ημ^2αημ^2βημ^2γ = συν^2ασυν^2βσυν^2γ$

ἢ $\eta\mu^2\beta\eta\mu^2\gamma(1-\eta\mu^2\alpha)+\eta\mu^2\gamma\eta\mu^2\alpha(1-\eta\mu^2\beta)+\eta\mu^2\alpha\eta\mu^2\beta(1-\eta\mu^2\gamma)+$
$$+2\eta\mu^2\alpha\eta\mu^2\beta\eta\mu^2\gamma=(1-\eta\mu^2\alpha)(1-\eta\mu^2\beta)(1-\eta\mu^2\gamma)$$
καὶ μετὰ τὰς πράξεις καὶ ἀναγωγὰς λαμβάνομεν: $\eta\mu^2\alpha+\eta\mu^2\beta+\eta\mu^2\gamma=1.$

Διὰ νὰ ὑπάρχουν αἱ γωνίαι α, β, γ, πρέπει: $\sigma\upsilon\nu A\sigma\upsilon\nu B\sigma\upsilon\nu\Gamma\geqslant 0.$

Πῶς θὰ ἀποδειχθῇ τὸ τελευταῖον τοῦτο ἐρώτημα;

139. Ἐὰν $\eta\mu^2x\eta\mu^2y+\eta\mu^2(x+y)=(\eta\mu x+\eta\mu y)^2$, τότε τὸ ἓν τῶν τόξων x καὶ y εἶναι πολ/σιον τοῦ π.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται διαδοχικῶς:
$$\eta\mu^2x\eta\mu^2y+(\eta\mu x\sigma\upsilon\nu y+\eta\mu y\sigma\upsilon\nu x)^2-\eta\mu^2x-\eta\mu^2y-2\eta\mu x\eta\mu y=0$$
$$\eta\mu^2x\eta\mu^2y+\eta\mu^2x\sigma\upsilon\nu^2y+\eta\mu^2y\sigma\upsilon\nu^2x+2\eta\mu x\eta\mu y\sigma\upsilon\nu x\sigma\upsilon\nu y-\eta\mu^2x-$$
$$-\eta\mu^2y-2\eta\mu x\eta\mu y=0.$$
$$\eta\mu^2x\eta\mu^2y-\eta\mu^2x(1-\sigma\upsilon\nu^2y)-\eta\mu^2y(1-\sigma\upsilon\nu^2x)+2\eta\mu x\eta\mu y\sigma\upsilon\nu x\sigma\upsilon\nu y-2\eta\mu x\eta\mu y=0$$
$$\eta\mu^2x\eta\mu^2y-\eta\mu^2x\eta\mu^2y-\eta\mu^2y\eta\mu^2x+2\eta\mu x\eta\mu y\sigma\upsilon\nu x\sigma\upsilon\nu y-2\eta\mu x\eta\mu y=0$$
ἢ $$\eta\mu x\eta\mu y(2\sigma\upsilon\nu x\sigma\upsilon\nu y-\eta\mu x\eta\mu y-2)=0,$$
ἢ $$\eta\mu x\eta\mu y[\sigma\upsilon\nu(x+y)+\sigma\upsilon\nu x\sigma\upsilon\nu y-2]=0.$$

Ἡ ἐντὸς τῆς ἀγκύλης ποσότης μηδενίζεται διὰ
$$\sigma\upsilon\nu(x+y)=1 \quad \text{καὶ} \quad \sigma\upsilon\nu x\sigma\upsilon\nu y=1.$$
Ἄρα $\eta\mu x\eta\mu y=0$, ὁπότε

ἢ $\eta\mu x=0 \Longrightarrow x=k\pi$ ἢ $\eta\mu y=0 \Longrightarrow y=k_1\pi.$

140. Εἰς πᾶν τρίγωνον ΑΒΓ νὰ ἀποδειχθῇ ὅτι:
$$\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma\geqslant\sqrt{3}.$$

Λύσις. Ἐπειδὴ $A+B+\Gamma=\pi \Longrightarrow A=\pi-(B+\Gamma)$

ἢ $$\sigma\varphi A=\sigma\varphi[\pi-(A+B)]=-\sigma\varphi(B+\Gamma)=-\frac{\sigma\varphi B\sigma\varphi\Gamma-1}{\sigma\varphi B+\sigma\varphi\Gamma} \qquad (1)$$

Ὑποθέτομεν $B<90^\circ$, $\Gamma<90^\circ$, καθόσον δύο γωνίαι τοῦ τριγώνου ΑΒΓ δύνανται νὰ εἶναι ὀξεῖαι, καὶ θέτομεν
$$\sigma\varphi B=x, \quad \sigma\varphi\Gamma=y \qquad (2)$$
καὶ ἡ (1) γίνεται:
$$\sigma\varphi A=\frac{xy-1}{x+y}, \text{ ὁπότε θὰ εἶναι :}$$
$$\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma=\frac{1-xy}{x+y}+x+y=\frac{(x+y)^2-xy+1}{x+y} \qquad (3)$$
Ἐκ τῶν (2) φαίνεται ὅτι $x>0$, $y>0$ καὶ ἄρα $x+y>0.$

Ἀρκεῖ νὰ δειχθῇ τώρα ὅτι
$$\frac{(x+y)^2-xy+1}{x+y}\geqslant\sqrt{3} \ (4) \quad \text{ἢ} \quad (x+y)^2-xy+1\geqslant\sqrt{3}(x+y)$$
ἢ $$x^2+y^2+xy+1-\sqrt{3}x-\sqrt{3}y\geqslant 0 \quad \text{ἢ} \quad x^2+(y-\sqrt{3})x+y^2-y\sqrt{3}+1\geqslant 0 \qquad (5)$$
Ἡ διακρίνουσα τοῦ τριωνύμου τοῦ πρώτου μέλους τῆς (5) εἶναι:
$$\Delta=(y-\sqrt{3})^2-4(y^2-y\sqrt{3}+1)=-(3y^2-2y\sqrt{3}+1)=-(y\sqrt{3}-1)^2\leqslant 0$$

$\left(\text{καὶ μηδενίζεται διὰ } y=\frac{\sqrt{3}}{3}, \text{ ὅτε καὶ } x=\frac{\sqrt{3}}{3}\right)$. Ἄρα ἰσχύει ἡ (5).

Ἄρα καὶ ἡ (4), δηλαδὴ $\sigma\varphi A+\sigma\varphi B+\sigma\varphi\Gamma \geqslant \sqrt{3}$.

Τὸ = ἰσχύει ὅταν $B=\Gamma=A=\frac{\pi}{3}$.

141. Ἐὰν $0\leqslant\alpha<\frac{\pi}{2}$ καὶ $0\leqslant\beta<\frac{\pi}{2}$, νὰ ἀποδειχθῇ ὅτι:

$$\varepsilon\varphi\frac{\alpha+\beta}{2}<\frac{1}{2}(\varepsilon\varphi\alpha+\varepsilon\varphi\beta)$$

ἂν $\alpha\neq\beta$ καὶ ὄχι συγχρόνως μηδέν.

Λύσις. Θέτομεν $\varepsilon\varphi\frac{\alpha}{2}=\lambda\ (0\leqslant\lambda<1)$ καὶ $\varepsilon\varphi\frac{\beta}{2}=\mu\ (0\leqslant\mu<1)$,

Τότε θὰ εἶναι:

$$\varepsilon\varphi\left(\frac{\alpha+\beta}{2}\right)=\frac{\varepsilon\varphi\frac{\alpha}{2}+\varepsilon\varphi\frac{\beta}{2}}{1\quad \varepsilon\varphi\frac{\alpha}{2}\,\varepsilon\varphi\frac{\beta}{2}}=\frac{\lambda+\mu}{1-\lambda\mu}\quad\text{καὶ}$$

$$\frac{1}{2}\left(\varepsilon\varphi\alpha+\varepsilon\varphi\beta\right)=\frac{1}{2}\left[\frac{2\varepsilon\varphi\frac{\alpha}{2}}{1-\varepsilon\varphi^2\frac{\alpha}{2}}+\frac{2\varepsilon\varphi\frac{\beta}{2}}{1-\varepsilon\varphi^2\frac{\beta}{2}}\right]=\frac{1}{2}\left[\frac{2\lambda}{1-\lambda^2}+\frac{2\mu}{1-\mu^2}\right]=$$

$$=\frac{\lambda(1-\mu^2)+\mu(1-\lambda^2)}{(1-\lambda^2)(1-\mu^2)}=\frac{(\lambda+\mu)(1-\lambda\mu)}{(1-\lambda^2)(1-\mu^2)}.$$

Ἀρκεῖ λοιπὸν νὰ δειχθῇ ὅτι: $\frac{\lambda+\mu}{1-\lambda\mu}<\frac{(\lambda+\mu)(1-\lambda\mu)}{(1-\lambda^2)(1-\mu^2)}$

Ἀλλὰ $\lambda+\mu\geqslant0$. Ἄρα $\frac{1}{1-\lambda\mu}\leqslant\frac{1-\lambda\mu}{(1-\lambda^2)(1-\mu^2)}$

ἢ
$$(1-\lambda^2)(1-\mu^2)\leqslant(1-\lambda\mu)^2. \qquad (1)$$

Ἐπειδὴ $1-\lambda\mu>0$ καὶ $(1-\lambda^2)(1-\mu^2)>0$, ἡ τελευταία σχέσις (1) γράφεται:

$-\lambda^2-\mu^2\leqslant 2\lambda\mu$ ἢ $(\lambda-\mu)^2\geqslant0$, ἥτις ἰσχύει διὰ κάθε λ,μ.

Ἄρα ἰσχύει καὶ ἡ $\varepsilon\varphi\frac{\alpha+\beta}{2}<\frac{1}{2}(\varepsilon\varphi\alpha+\varepsilon\varphi\beta)$.

142. Ἐὰν εἰς τρίγωνον ΑΒΓ ἀληθεύῃ ἡ ἰσότης

$$\frac{\eta\mu^2B}{\eta\mu^2\Gamma}-\frac{\sigma\upsilon\nu^2B}{\sigma\upsilon\nu^2\Gamma}=\frac{\beta^4-\gamma^4}{\beta^2\gamma^2},$$

νὰ ἀποδειχθῇ ὅτι ἢ $B=\Gamma$ ἢ $A=90^\circ$ ἢ $|B-\Gamma|=\frac{\pi}{2}$.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$\frac{\eta\mu^2B}{\eta\mu^2\Gamma}-\frac{\sigma\upsilon\nu^2B}{\sigma\upsilon\nu^2\Gamma}=\frac{\eta\mu^4B-\eta\mu^4\Gamma}{\eta\mu^2B\eta\mu^2\Gamma}=\frac{\eta\mu^2B}{\eta\mu^2\Gamma}-\frac{\eta\mu^2\Gamma}{\eta\mu^2B}$$

ἢ $\frac{\sigma\upsilon\nu^2B}{\sigma\upsilon\nu^2\Gamma}=\frac{\eta\mu^2\Gamma}{\eta\mu^2B}$ ἢ $\eta\mu^2B\sigma\upsilon\nu^2B=\eta\mu^2\Gamma\sigma\upsilon\nu^2\Gamma$

ἢ $4\eta\mu^2 B\sigma\upsilon\nu^2 B = 4\eta\mu^2\Gamma\sigma\upsilon\nu^2\Gamma$ ἢ $\eta\mu^2 2B = \eta\mu^2 2\Gamma$

ἢ $(\eta\mu 2B + \eta\mu 2\Gamma)(\eta\mu 2B - \eta\mu 2\Gamma) = 0$

ἐξ οὗ: ἢ $\eta\mu 2B = \eta\mu 2\Gamma$ ἢ $\eta\mu 2B = -\eta\mu 2\Gamma$

1ον: Ἔστω $\eta\mu 2B = \eta\mu 2\Gamma$, ὅτε $2B = 2\Gamma \Longrightarrow B = \Gamma$

$2B + 2\Gamma = \pi \Longrightarrow B + \Gamma = \frac{\pi}{2} \Longrightarrow A = 90^{\circ}$

2ον: Ἔστω $\eta\mu 2B = -\eta\mu 2\Gamma$, ὁπότε

$$|2B - 2\Gamma| = \pi \quad ἢ \quad |B - \Gamma| = \frac{\pi}{2}.$$

Εἰς τὴν περίπτωσιν ταύτην τὸ τρίγωνον ΑΒΓ καλεῖται ψευδοορθογώνιον. Ὥστε θὰ εἶναι:

ἢ $B = \Gamma$ ἢ $A = 90^{\circ}$ ἢ $|B - \Gamma| = \frac{\pi}{2}$.

143. Παρατηροῦντες ὅτι αἱ γωνίαι $\frac{\pi}{7}$, $\frac{2\pi}{7}$, $\frac{4\pi}{7}$ δύναται νὰ θεωρηθοῦν ὡς γωνίαι ἑνὸς τριγώνου, νὰ ἀποδειχθῇ ὅτι:

$$\sigma\upsilon\nu\frac{\pi}{7}\,\sigma\upsilon\nu\frac{2\pi}{7}\,\sigma\upsilon\nu\frac{4\pi}{7} = -\frac{1}{8}.$$

Λύσις. Ἔχομεν: $\frac{\pi}{7} + \frac{2\pi}{7} + \frac{4\pi}{7} = \pi$. Ἐὰν α, β, γ εἶναι αἱ πλευραὶ τοῦ τριγώνου ΑΒΓ μὲ $A = \frac{\pi}{7}$, $B = \frac{2\pi}{7}$ καὶ $\Gamma = \frac{4\pi}{7}$, τότε

$$\frac{\eta\mu\frac{\pi}{7}}{\alpha} = \frac{\eta\mu\frac{2\pi}{7}}{\beta} = \frac{\eta\mu\frac{4\pi}{7}}{\gamma}.$$

Ἐκ τῆς $\frac{\eta\mu\frac{\pi}{7}}{\alpha} = \frac{\eta\mu\frac{2\pi}{7}}{\beta} = \frac{2\eta\mu\frac{\pi}{7}\,\sigma\upsilon\nu\frac{\pi}{7}}{\beta} \Longrightarrow 2\sigma\upsilon\nu\frac{\pi}{7} = \frac{\beta}{\alpha}$ (1)

Ἐκ τῆς $\frac{\eta\mu\frac{2\pi}{7}}{\beta} = \frac{\eta\mu\frac{4\pi}{7}}{\gamma} = \frac{2\eta\mu\frac{2\pi}{7}\,\sigma\upsilon\nu\frac{2\pi}{7}}{\gamma} \Longrightarrow 2\sigma\upsilon\nu\frac{2\pi}{7} = \frac{\gamma}{\alpha}$ (2)

Ἀλλὰ $\eta\mu\frac{\pi}{7} = -\eta\mu\left(\pi + \frac{\pi}{7}\right) = -\eta\mu\frac{8\pi}{7}$, ὁπότε

$$\frac{-\eta\mu\frac{8\pi}{7}}{\alpha} = \frac{\eta\mu\frac{4\pi}{7}}{\gamma} \quad ἢ \quad \frac{-2\eta\mu\frac{4\pi}{7}\,\sigma\upsilon\nu\frac{4\pi}{7}}{\alpha} = \frac{\eta\mu\frac{4\pi}{7}}{\gamma}$$

ἐξ οὗ: $2\sigma\upsilon\nu\frac{4\pi}{7} = -\frac{\alpha}{\gamma}$,

Κατ' ἀκολουθίαν:

$$2^3\cdot\sigma\upsilon\nu\frac{\pi}{7}\,\sigma\upsilon\nu\frac{2\pi}{7}\,\sigma\upsilon\nu\frac{4\pi}{7} = \frac{\beta}{\alpha}\cdot\frac{\gamma}{\beta}\left(-\frac{\alpha}{\gamma}\right) = -1$$

ἢ $\sigma\upsilon\nu\frac{\pi}{7}\,\sigma\upsilon\nu\frac{2\pi}{7}\,\sigma\upsilon\nu\frac{4\pi}{7} = -\frac{1}{8}$.

144. Ἐὰν εἰς τρίγωνον ΑΒΓ ἀληθεύῃ ἡ ἰσότης ημ4Α+ημ4Β+ημ4Β=0, νὰ ἀποδειχθῇ ὅτι τοῦτο εἶναι ὀρθογώνιον.

Λύσις. Ἡ δοθεῖσα σχέσις γράφεται:

$$2\eta\mu(2A+2B)\sigma\upsilon\nu(2A-2B)+2\eta\mu2\Gamma\sigma\upsilon\nu2\Gamma=0$$

ἢ
$$-\eta\mu2\Gamma\sigma\upsilon\nu(2A-2B)+\eta\mu2\Gamma\sigma\upsilon\nu2\Gamma=0$$

ἢ
$$-\eta\mu2\Gamma\cdot[\sigma\upsilon\nu(2A-2B)-\sigma\upsilon\nu(2A+2B)]=0$$

ἢ
$$\eta\mu2\Gamma\cdot2\eta\mu2A\eta\mu2B=0$$

ἢ
$$\eta\mu2A\eta\mu2B\eta\mu2\Gamma=0.$$

Ὅθεν
$$\left.\begin{array}{llll} \text{ἢ} & \eta\mu2A=0 \Longrightarrow & 2A=180^0 \Longrightarrow & A=90^0 \\ \text{ἢ} & \eta\mu2B=0 \Longrightarrow & 2B=180^0 \Longrightarrow & B=90^0 \\ \text{ἢ} & \eta\mu2\Gamma=0 \Longrightarrow & 2\Gamma=180^0 \Longrightarrow & \Gamma=90^0 \end{array}\right\}.$$

145. Ἀφοῦ ἀποδειχθῇ ἡ ταυτότης εφx=σφx−2σφ2x, νὰ ἀποδειχθῇ ἀκολούθως ὅτι:

$$S_\nu=\frac{1}{2}\varepsilon\varphi\frac{x}{2}+\frac{1}{2^2}\varepsilon\varphi\frac{x}{2^2}+\cdots+\frac{1}{2^\nu}\varepsilon\varphi\frac{x}{2^\nu}=\frac{1}{2^\nu}\sigma\varphi\frac{x}{2^\nu}-\sigma\varphi x$$

ἔνθα $0<x<\frac{\pi}{2}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$\varepsilon\varphi x=\frac{\eta\mu x}{\sigma\upsilon\nu x}=\frac{\eta\mu^2 x}{\eta\mu x\sigma\upsilon\nu x}=\frac{\sigma\upsilon\nu^2 x-(\sigma\upsilon\nu^2 x-\eta\mu^2 x)}{\eta\mu x\sigma\upsilon\nu x}=$$

$$=\frac{\sigma\upsilon\nu^2 x-\sigma\upsilon\nu2x}{\eta\mu x\sigma\upsilon\nu x}=\frac{\sigma\upsilon\nu^2 x}{\eta\mu x\sigma\upsilon\nu x}-\frac{\sigma\upsilon\nu2x}{\eta\mu x\sigma\upsilon\nu x}=$$

$$=\frac{\sigma\upsilon\nu x}{\eta\mu x}-\frac{2\sigma\upsilon\nu2x}{2\eta\mu x\sigma\upsilon\nu x}=\sigma\varphi x-\frac{2\sigma\upsilon\nu2x}{\eta\mu2x}=\sigma\varphi x-2\sigma\varphi2x.$$

Ὥστε:
$$\varepsilon\varphi x=\sigma\varphi x-2\sigma\varphi2x. \qquad (1)$$

Εἰς τὴν (1) θέτομεν ἀντὶ x, τὸ $\frac{x}{2}$, $\frac{x}{2^2}$, $\frac{x}{2^3}$, ..., $\frac{x}{2^{\nu-1}}$ καὶ ἔχομεν ἀντιστοίχως:

$$\varepsilon\varphi\frac{x}{2}=\sigma\varphi\frac{x}{2}-2\sigma\varphi x$$
$$\varepsilon\varphi\frac{x}{2^2}=\sigma\varphi\frac{x}{2^2}-2\sigma\varphi\frac{x}{2}$$
$$\cdots\cdots\cdots$$
$$\varepsilon\varphi\frac{x}{2^{\nu-1}}=\sigma\varphi\frac{x}{2^{\nu-1}}-2\sigma\varphi\frac{x}{2^{\nu-2}}$$
$$\varepsilon\varphi\frac{x}{2^\nu}=\sigma\varphi\frac{x}{2^\nu}-2\sigma\varphi\frac{x}{2^{\nu-1}}$$

$$\Longrightarrow \left.\begin{array}{l} \frac{1}{2}\varepsilon\varphi\frac{x}{2}=\frac{1}{2}\sigma\varphi\frac{x}{2}-\sigma\varphi x \\ \frac{1}{2^2}\varepsilon\varphi\frac{x}{2^2}=\frac{1}{2^2}\sigma\varphi\frac{x}{2^2}-\frac{1}{2}\sigma\varphi\frac{x}{2} \\ \cdots\cdots\cdots \\ \frac{1}{2^{\nu-1}}\varepsilon\varphi\frac{x}{2^{\nu-1}}=\frac{1}{2^{\nu-1}}\sigma\varphi\frac{x}{2^{\nu-1}}- \\ \qquad -\frac{1}{2^{\nu-2}}\sigma\varphi\frac{x}{2^{\nu-2}} \\ \frac{1}{2^\nu}\varepsilon\varphi\frac{x}{2^\nu}=\frac{1}{2^\nu}\sigma\varphi\frac{x}{2^\nu}- \\ \qquad -\frac{1}{2^\nu}\sigma\varphi\frac{x}{2^{\nu-1}} \end{array}\right\} \quad (2)$$

Διὰ προσθέσεως κατὰ μέλη τῶν (2) λαμβάνομεν :

$$S_\nu = \frac{1}{2}\,\text{εφ}\,\frac{x}{2} + \frac{1}{2^2}\,\text{εφ}\,\frac{x}{2^2} + \ldots + \frac{1}{2^\nu}\,\text{εφ}\,\frac{x}{2^\nu} = \frac{1}{2^\nu}\,\text{σφ}\,\frac{x}{2^\nu} - \text{σφ}x.$$

146. Νὰ ἀποδειχθῇ ὅτι ὑφίστανται δύο ἀριθμοὶ x καὶ y, τοιοῦτοι ὥστε: $\text{στεμ}\alpha = x\,\text{σφ}\,\frac{\alpha}{2} + y\,\text{σφ}\alpha$, οἱουδήποτε ὄντος τοῦ α. Ἀκολούθως δείξατε ὅτι :

$$S_\nu = \text{στεμ}\alpha + \text{στεμ}2\alpha + \text{στεμ}4\alpha + \cdots + \text{στεμ}2^\nu\alpha = \text{σφ}\frac{\alpha}{2} - \text{σφ}2^\nu\alpha.$$

Λύσις. Ἡ δοθεῖσα σχέσις γάρφεται :

$$\frac{1+\text{εφ}^2\frac{\alpha}{2}}{2\text{εφ}\,\frac{\alpha}{2}} = x\cdot\frac{1}{\text{εφ}\,\frac{\alpha}{2}} + y\cdot\frac{1-\text{εφ}^2\frac{\alpha}{2}}{2\text{εφ}\,\frac{\alpha}{2}}$$

ἐξ οὗ : $$(y+1)\,\text{εφ}^2\frac{\alpha}{2} + 1 - 2x - y = 0.$$

Ἀφοῦ ἡ σχέσις αὕτη ἰσχύει διὰ πᾶσαν τιμὴν τοῦ α, ἄρα καὶ διὰ πᾶσαν τιμὴν τῆς $\text{εφ}\frac{\alpha}{2}$, θὰ ἔχωμεν :

$$\left.\begin{aligned} y+1&=0 \\ 1-2x-y&=0 \end{aligned}\right\} \Longrightarrow \left.\begin{aligned} x&=1 \\ y&=-1 \end{aligned}\right\},$$

ὁπότε ἡ δοθεῖσα σχέσις γράφεται :

$$\text{στεμ}\alpha = \text{σφ}\frac{\alpha}{2} - \text{σφ}\alpha \qquad (1)$$

καὶ ἄρα : $$\text{στεμ}2\alpha = \text{σφ}\alpha - \text{σφ}2\alpha$$

. . . .

$$\text{στεμ}2^{\nu-1}\alpha = \text{σφ}2^{\nu-2}\alpha - \text{σφ}2^{\nu-1}\alpha.$$

Διὰ προσθέσεως τούτων κατὰ μέλη, λαμβάνομεν

$$S_\nu = \text{στεμ}\alpha + \text{στεμ}2\alpha + \text{στεμ}4\alpha + \cdots + \text{στεμ}2^\nu\alpha = \text{σφ}\,\frac{\alpha}{2} - \text{σφ}2^{\nu-1}\alpha.$$

ΛΟΓΑΡΙΘΜΙΚΑΙ ΑΣΚΗΣΕΙΣ

147. Νὰ εὑρεθοῦν οἱ λογάριθμοι τῶν τριγωνομετρικῶν ἀριθμῶν:

1. **ημ(15°27′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\eta\mu(15^\circ 27') = \overline{1},42553$.

2. **συν(36°12′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\sigma\upsilon\nu(36^\circ 12') = \overline{1},90685$.

3. **συν(65°25′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\sigma\upsilon\nu(65^\circ 25') = \overline{1},61911$.

4. **ημ(58°10′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\eta\mu(58^\circ 10') = \overline{1},92921$.

5. **εφ(20°16′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\varepsilon\varphi(20^\circ 16') = \overline{1},56732$.

6. **εφ(53°6′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\varepsilon\varphi(53^\circ 6') = 0,12446$.

7. **σφ(14°36′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\sigma\varphi(14^\circ 36') = 0,58422$.

8. **σφ(70°14′).**

Λύσις. Ἔχομεν: $\lambda o\gamma\sigma\varphi(70^\circ 14') = \overline{1},55554$.

9. **ημ(25°10′18″).**

Λύσις. Διάταξις τῶν πράξεων:

$\lambda o\gamma\eta\mu(25^\circ 11') = \overline{1},62892$	60″	27	μ.έ.δ.τ.
$\lambda o\gamma\eta\mu(25^\circ 10') = \overline{1},62865$	18	x ;	
$\Delta = \quad 27$	$x = 27 \cdot \dfrac{18}{60} = 8$ μ.έ δ.τ.		

Ἄρα: $\lambda o\gamma\eta\mu(25^\circ 10' 18'') = \overline{1},62865 + 0,00008 = \overline{1},62873$.

10. **ημ(55°26′39″).**

Λύσις. Ἔχομεν:

$\lambda o\gamma\eta\mu(55^\circ 27') = \overline{1},91573$	60″	8	μ.έ.δ.τ.
$\lambda o\gamma\eta\mu(55^\circ 26') = \overline{1},91565$	39″	x ;	
$\Delta = \quad 8$	$x = 8 \cdot \dfrac{39}{60} = 5$ μ.έ.δ.τ.		

Ἄρα: $\lambda o\gamma\eta\mu(55^\circ 26' 39'') = \overline{1},91568 + 0,00005 = \overline{1},91570$,

11. **συν(33°17′25″)**

Λύσις. Διάταξις τῶν πράξεων:

$\lambda o\gamma\sigma\upsilon\nu(33^\circ 17') = \overline{1},92219$	60″	8	μ.έ δ.τ.
$\lambda o\gamma\sigma\upsilon\nu(33^\circ 18') = \overline{1},92211$	25″	x ;	
$\Delta = \quad 8$	$x = 8 \cdot \dfrac{25}{60} = \dfrac{200}{60} = \dfrac{10}{3} = 3$ μ.έ.δ.τ.		

Ἄρα : λογσυν(33°17′25′′) = $\overline{1}$,92219 — 0,00003 = $\overline{1}$,92216.

12. **συν(66°14′52′′).**

Λύσις. Διάταξις τῶν πράξεων:

λογσυν(66°14′) = $\overline{1}$,60532	60′′ 29 μ.έ.δ.τ.
λογσυν(66°15′) = $\overline{1}$,60503	52′′ x ;
Δ = 29	$x = 29 \cdot \frac{52}{60} = 25$ μ.έ.δ.τ.

Ἄρα : λογσυν(66°14 52′′) = $\overline{1}$,60532 − 0,00025 = $\overline{1}$,60507.

13. **εφ(18°56′10′′).**

Λύσις. Διάταξις τῶν πράξεων:

λογεφ(18°57′) = $\overline{1}$,53574	60′′ 41 μ.έ.δ.τ.
λογεφ(18°56′) = $\overline{1}$,53533	10′′ x ;
Δ = 41	$x = 41 \cdot \frac{10}{60} = \frac{41}{6} = 6{,}5$ ἢ 7 μ.έ.δ.τ.

Ἄρα : λογεφ(18°56′10′′) = $\overline{1}$,53533 + 0,00007 = $\overline{1}$,53540.

14. **εφ(48°10′50′′).**

Λύσις. Διάταξις τῶν πράξεων:

λογεφ(48°11′) = 0,04836	60′′ 26 μ.έ.δ.τ.
λογεφ(48°10′) = 0,04810	50′′ x ;
Δ = 26	$x = 26 \cdot \frac{50}{60} = \frac{130}{6} = 2{,}16$ ἢ 2 μ.έ.δ.τ.

Ἄρα : λογεφ(48°15′50′′) = $\overline{1}$,04810 + 0,00002 = 0,04812.

15. **σφ(29°33′48′′).**

Λύσις. Διάταξις τῶν πράξεων:

λογσφ(29°33′) = 0,24647	60′ 29 μ.έ.δ.τ.
λογσφ(29°34′) = 0,24618	48′′ x ;
Δ = 29	$x = 29 \cdot \frac{48}{60} = 29 \cdot \frac{4}{5} = \frac{116}{5} = 23{,}2$ ἢ 23 μ.έ.δ.τ.

Ἄρα : λογσφ(29°33′48′′) = 0,24647 — 0,00023 = 0,24624.

16. **σφ(24°19′10′′).**

Λύσις. Διάταξις τῶν πράξεων:

λογσφ(24°19′) = 0,34499	60′′ 34 μ.έ.δ.τ.
λογσφ(21°20′) = 0,34465	10′′ x ;
Δ = 34	$x = 34 \cdot \frac{10}{60} = \frac{34}{6} = 5{,}66$ ἢ 6 μ.έ.δ.τ.

Ἄρα : λογσφ(24°19′10′′) = 0,34499 − 0,00006 = 0,34493.

17. **σφ(70°64′15″).**

Λύσις. Διάταξις τῶν πράξεων :

λογσφ(70°34′) = $\overline{1}$,54754	60″ 40 μ.έ.δ.τ.
γογσφ(70°35′) = $\overline{1}$,54714	15″ x ;
Δ = 40	$x = 40 \cdot \frac{15}{60} = \frac{40}{4} = 10$ μ.έ.δ.τ.

Ἄρα : λογσφ(70°34′15″) = $\overline{1}$,54754 − 0,00010 = $\overline{1}$,54744.

18. **ημ(123°56′10″).**

Λύσις. Διάταξις τῶν πράξεων. Εἶναι :

ημ(123°56′10″) = ημ(179°59′60″ — 123°56′10″) = ημ(76°3′50″).

λογημ(76°4′) = $\overline{1}$,98703	60″ 3 μ.έ.δ.τ.
λογημ(67°3′) = $\overline{1}$,98700	50″ x ;
Δ = 3	$x = 3 \cdot \frac{50}{60} = \frac{5}{2} = 2{,}5$ ἢ 3 μ.έ.δ.τ.

Ἄρα : λογημ(123°56′10″) = λογημ(76°3′50″)

= $\overline{1}$,98700 + 0,00003 = $\overline{1}$,98703.

148. Νὰ εὑρεθῇ ὁ λογάριθμος τῶν τριγωνομετρικῶν ἀριθμῶν.

1. $ημ \dfrac{3π}{7}$.

Λύσις. Εἶναι : $ημ \dfrac{3π}{7}$ = ημ(77°8′34″,28).

Διάταξις τῶν πράξεων :

λογημ(77°9′) = $\overline{1}$,98898	60″ 2 μ.έ.δ.τ
λογημ(77°8′) = $\overline{1}$,98896	34″,28 x ;
Δ = 2	$x = 2 \cdot \frac{34{,}28}{60} = \frac{34{,}28}{30} = 1{,}14$ ἢ 1 μ.έ.δ.τ.

Ἄρα : λογημ $\dfrac{3π}{7}$ = λογημ(77°8′34″,28).

= $\overline{1}$,98896 + 0,00001 = $\overline{1}$,98897.

2. $συν \dfrac{π}{17}$.

Λύσις. Εἶναι : $συν \dfrac{π}{17}$ = συν(10°35′17″,6).

λογσυν(10°35′) = $\overline{1}$,99255	60″ 3 μ.έ.δ,τ.
λογσυν(10°36′) = $\overline{1}$,99252	17″,6 x ;
Δ = 3	$x = 3 \cdot \frac{17{,}6}{60} = \frac{17{,}6}{20} = 0{,}8$ ἢ 1 μ.έ.δ.τ.

Ἄρα : λογσυν $\dfrac{π}{17}$ = λογσυν(10°35′17″,6)

= $\overline{1}$,99255 − 0,00001 = $\overline{1}$,99254.

3. $\varepsilon\varphi \, \dfrac{3\pi}{11}$.

Λύσις. Εἶναι : $\varepsilon\varphi \, \dfrac{3\pi}{11} = \varepsilon\varphi(49^\circ 5' 27'',27)$.

$\lambda o\gamma\varepsilon\varphi(49^\circ 6') = 0{,}06237$	$60''$ 26 μ.έ.δ.τ.
$\lambda o\gamma\varepsilon\varphi(49^\circ 5') = 0{,}06211$	$27'',27$ x ;
$\Delta =$ 26	$x = 26 \cdot \dfrac{27{,}27}{60} = 10{,}48$ ἢ 10 μ.έ.δ.τ.

Ἄρα : $\lambda o\gamma\varepsilon\varphi \, \dfrac{3\pi}{11} = \lambda o\gamma\varepsilon\varphi(49^\circ 5' 27'',27) = 0{,}06211 + 0{,}00010 = 0{,}06221$.

4. $\sigma\varphi \, \dfrac{5\pi}{17}$.

Λύσις. Εἶναι : $\sigma\varphi \, \dfrac{5\pi}{17} = \sigma\varphi(52^\circ 56' 28'',23)$

$\lambda o\gamma\sigma\varphi(52^\circ 26') = \overline{1},88603$	$60''$ 26 μ.έ.δ.τ.
$\lambda o\gamma\sigma\varphi(52^\circ 27') = \overline{1},88577$	$28'',23$ x ;
$\Delta =$ 26	$x = 26 \cdot \dfrac{28{,}23}{60} = \dfrac{366{,}99}{30} = 12{,}23$ ἢ 12 μ.έ.δ.τ.

Ἄρα : $\lambda o\gamma\sigma\varphi \, \dfrac{5\pi}{17} = \lambda o\gamma\sigma\varphi(52^\circ 56' 28'',23) = \overline{1},88603 - 0{,}00012 = \overline{1},88591$.

149. Νὰ εὑρεθοῦν αἱ μεταξὺ 0° καὶ 90° τιμαὶ τοῦ τόξου x, αἱ ὁποῖαι ἱκανοποιοῦν τὰς ἐξισώσεις :

1. **$\lambda o\gamma\eta\mu x = \overline{1},84439$** ***Λύσις.*** $x = 44^\circ 20' 9''$.
2. **$\lambda o\gamma\sigma\upsilon\nu x = \overline{1},65190$** » $x = 63^\circ 20' 36''$.
3. **$\lambda o\gamma\varepsilon\varphi x = \overline{1},26035$** » $x = 10^\circ 19' 17''$.
4. **$\lambda o\gamma\sigma\varphi x = \overline{1},59183$** » $x = 68^\circ 39' 36''$.
5. **$\lambda o\gamma\sigma\varphi x = 0{,}21251$** » $x = 31^\circ 30' 36''$.
6. **$\lambda o\gamma\varepsilon\varphi x = \overline{1},18954$** » $x = 57^\circ 7' 25'',71$.
7. **$\lambda o\gamma\tau\varepsilon\mu x = 0{,}02830$** » $\lambda o\gamma\sigma\upsilon\nu x = \overline{1},97170 \Longrightarrow x = 20^\circ 27' 36''$.

Πρὸς λύσιν τῶν ἀνωτέρω ἀσκήσεων θὰ στηριχθῆτε εἰς τὰ παραδείγματα τοῦ βιβλίου καὶ εἰς τὸ σχετικὸν κεφάλαιον.

150. Νὰ ὑπολογισθοῦν αἱ μικρότεραι θετικαὶ τιμαὶ τοῦ τόξου x, αἱ ὁποῖαι εἶναι ρίζαι τῶν ἀκολούθων ἐξισώσεων :

1. $\eta\mu x = -\dfrac{3}{5}$.

Λύσις. Εἶναι : $-\eta\mu x = \dfrac{3}{5} = 0{,}6$ ἢ $\eta\mu(-x) = 0{,}6$

ἢ $\lambda o\gamma\eta\mu(-x) = \lambda o\gamma\, 0{,}6 = \overline{1},77815$, ἐξ οὗ :

$$-x = k \cdot 360^\circ + 36^\circ 52' 11'' \quad \text{ἢ} \quad 143^\circ 7' 49''$$
$$x = k \cdot 360^\circ - 36^\circ 52' 11'' \quad \text{ἢ} \quad -143^\circ 7' 49''$$

Ἄρα, ἡ μικροτέρα θετικὴ τιμὴ τοῦ x εἶναι : $x' = 216^\circ 52' 11''$.

2. $\sigma\upsilon\nu x = -0,7.$

Λύσις. Ἔχομεν: $-\sigma\upsilon\nu x = 0,7$ ἢ $\sigma\upsilon\nu(180^0 - x) = 0,7$ ἢ
$\lambda o\gamma\sigma\upsilon\nu(180^0 - x) = \lambda o\gamma 0,7 = \overline{1},84510,$

καὶ ἄρα: $180^0 - x = k \cdot 360^0 \pm 45^0 34' 23''$
$x = k \cdot 360^0 + 134^0 25' 37''$ ἢ $225^0 34' 23''$.

Ἄρα, ἡ μικροτέρα θετικὴ τιμὴ τοῦ x εἶναι $x' = 134^0 25' 37''$.

3. $\varepsilon\varphi x = -3.$

Λύσις. Εἶναι: $-\varepsilon\varphi x = 3$ ἢ $\varepsilon\varphi(180^0 - x) = 3$ ἢ

ἢ $\lambda o\gamma\varepsilon\varphi(180^0 - x) = \lambda o\gamma 3 = 0,47712,$ ἐξ οὗ:
$180^0 - x = k \cdot 180^0 + 71^0 33' 54''$
$x = k \cdot 180^0 + 108^0 26' 6''$, $x' = 108^0 26' 6''$.

4. $\sigma\varphi x = \sigma\upsilon\nu 42^0.$

Λύσις. $\lambda o\gamma\sigma\varphi x = \lambda o\gamma\sigma\upsilon\nu 42^0 = \overline{1},87107$
$x = k \cdot 180^0 + 53^0 22' 58''$, ἢ $x' = 53^0 22' 58''$.

5. $\tau\varepsilon\mu x = -1,8.$

Λύσις. Εἶναι: $\sigma\upsilon\nu(180^0 - x) = \dfrac{10}{18}$ καὶ

$$\lambda o\gamma\sigma\upsilon\nu(180^0 - x) = \lambda o\gamma \frac{10}{18} = \lambda o\gamma 10 - \lambda o\gamma 18 = \overline{1},74473.$$

$180^0 - x = k \cdot 360^0 \pm 56^0 15' 3''$
$x = k \cdot 360^0 + 123^0 44' 57''$, ἢ $236^0 1' 53''$.

Ἄρα, ἡ μικροτέρα θετικὴ τιμὴ τοῦ x εἶναι $x' = 123^0 44' 57''$.

6. $\sigma\tau\varepsilon\mu x = -\dfrac{4}{3}.$

Λύσις. Εἶναι: $\dfrac{1}{\eta\mu x} = -\dfrac{4}{3}$ ἢ $\eta\mu x = -\dfrac{3}{4}$

ἢ $\eta\mu(-x) = \dfrac{3}{4} = 0,75$ καὶ $\lambda o\gamma\eta\mu(-x) = \lambda o\gamma 0,75 = \overline{1},87506$

ὁπότε: $-x = k \cdot 360^0 + 48^0 35' 25''$ ἢ $131^0 24' 35''$
$x = k \cdot 360^0 - 48^0 35' 25''$ ἢ $-131^0 24' 35''$.

Ἄρα, ἡ μικροτέρα θετικὴ τιμὴ τοῦ x εἶναι $x' = 228^0 35' 25''$.

7. $\sigma\upsilon\nu \dfrac{x}{2} = \varepsilon\varphi 150^0.$

Λύσις. Εἶναι: $\sigma\upsilon\nu \dfrac{x}{2} = \varepsilon\varphi 150^0 = -\varepsilon\varphi 30^0$ ἢ

$$\sigma\upsilon\nu\left(180^0 - \frac{x}{2}\right) = \varepsilon\varphi 30^0 \quad \text{καὶ} \quad \lambda o\gamma\sigma\upsilon\nu\left(180^0 - \frac{x}{2}\right) = \lambda o\gamma\varepsilon\varphi 30^0 = \overline{1},76144$$

ἢ $180^0 - \dfrac{x}{2} = k \cdot 360^0 \pm 54^0 44' 7''$ ἢ $x = (2k+1) 360^0 \pm 109^0 28' 14''$

καὶ $x' = 250^0 31' 46''$.

8. **$\eta\mu 2x = 0{,}58$.**

Λύσις. Εἶναι $\lambda o\gamma\eta\mu 2x = \lambda o\gamma 0{,}58 = \overline{1}{,}76343$

$$2x = k\cdot 360^0 + 35^0 27'3'' \quad ἤ \quad 144^0 32'57''$$

$$x = k\cdot 180^0 + 17^0 43'32'' \quad ἤ \quad 72^0 16'28''.$$

Ἡ μικροτέρα θετικὴ τιμὴ τοῦ x εἶναι $x' = 17^0 43'32''$.

9. $$\varepsilon\varphi\left(45^0 - \frac{x}{2}\right) = -\frac{17}{9}.$$

Λύσις. Εἶναι: $\varepsilon\varphi\left(135^0 - \frac{x}{2}\right) = \frac{17}{9}$ καὶ

$$\lambda o\gamma\varepsilon\varphi\left(135^0 - \frac{x}{2}\right) = \lambda o\gamma\frac{17}{9} = 0{,}27621$$

$$135^0 - \frac{x}{2} = k\cdot 180^0 + 62^0 6'10''$$

ἤ $$x = k\cdot 360^0 + 145^0 47'40'' \quad ἤ \quad x' = 145^0 47'40''.$$

151. Νὰ εὑρεθῇ τὸ ἐλάχιστον θετικὸν τόξον x, διὰ τὸ ὁποῖον εἶναι:

1. **$\lambda o\gamma\eta\mu x = \overline{3}{,}72835$.**

Λύσις. Ἐργαζόμενοι ὅπως καὶ εἰς τὰ προηγούμενα παραδείγματα, εὑρίσκομεν: $x = 18'23'',5$.

2. **$\lambda o\gamma\varepsilon\varphi x = \overline{2}{,}77213$.**

Λύσις. Εὑρίσκομεν εὐκόλως ὅτι: $x = 3^0 23'11'',4$.

3. **$\lambda o\gamma\sigma\varphi x = 1{,}53421$.**

Λύσις. Ἔχομεν:

$$\lambda o\gamma\sigma\varphi x = 1{,}53421 \Longrightarrow \lambda o\gamma\varepsilon\varphi x = \overline{2}{,}46579 \Longrightarrow x = 1^0 40'26'',9.$$

4. **$\lambda o\gamma\sigma\upsilon\nu x = \overline{2}{,}69231$.**

Λύσις. Εἶναι: $\lambda o\gamma\sigma\upsilon\nu x = \lambda o\gamma\eta\mu(90^0 - x) = \overline{2}{,}69231$

καὶ $$90^0 - x = 2^0 49'20'',5 \Longrightarrow x = 87^0 10'39'',5.$$

5. **$\lambda o\gamma\varepsilon\varphi x = 2{,}48739$.**

Λύσις. Εἶναι: $\lambda o\gamma\varepsilon\varphi(90^0 - x) = \overline{3}{,}51261$

$$90^0 - x = 11'11'',5 \Longrightarrow x = 88^0 48'48'',5.$$

6. **$\lambda o\gamma\sigma\varphi x = \overline{2}{,}53298$.**

Λύσις. Εἶναι: $\lambda o\gamma\varepsilon\varphi(90^0 - x) = \overline{2}{,}53298$

$$90^0 - x = 1^0 57'14'',7 \Longrightarrow x = 88^0 2'45'',3.$$

152. Νὰ εὑρεθῇ τὸ ἐλάχιστον θετικὸν τόξον x, διὰ τὸ ὁποῖον εἶναι:

$$\sigma\varphi x = \frac{\sqrt[3]{\alpha}\cdot\sigma\upsilon\nu A}{\eta\mu 5A\cdot\varepsilon\varphi B},$$

ἔνθα **$\alpha = -0{,}08562$, $A = 131^0 49'25''$, $B = 36^0 43'26''$.**

Λύσις. Κατὰ τὰ γνωστὰ εἶναι :

$\sqrt[3]{\alpha}=-\sqrt[3]{-\alpha}$, συνA=συν(131°49'25'')=−συν48°10'35'' καὶ ημ5A=−ημ(60°52'55).

Ἐὰν θέσωμεν α=48°10'25'', β=69°52'55'', τότε :

$$\sigma\varphi x=-\frac{\sqrt[3]{-\alpha\sigma\upsilon\nu\alpha}}{\eta\mu\beta\varepsilon\varphi B} \quad \text{ἢ} \quad \sigma\varphi(-x)=\frac{\sqrt[3]{-\alpha\sigma\upsilon\nu\alpha}}{\eta\mu\beta\varepsilon\varphi B}$$

$$\text{ἢ} \quad \lambda o\gamma\sigma\varphi(-x)=\frac{1}{3}\lambda o\gamma(-\alpha)+\lambda o\gamma\sigma\upsilon\nu\alpha+\sigma\upsilon\lambda o\gamma\cdot\eta\mu\beta+\sigma\upsilon\lambda o\gamma\varepsilon\varphi B.$$

Ἐὰν θ εἶναι ἡ μικροτέρα θετικὴ τιμὴ τοῦ τόξου −x, θὰ ἔχωμεν :

$$-x=k\cdot180^0+\theta \quad \text{ἢ} \quad x=k\cdot180^0-\theta.$$

Ὑπολογισμὸς τοῦ **x.** Κατὰ τὰ γνωστὰ εἶναι :

$$\frac{1}{3}\lambda o\gamma(-\alpha)=\frac{1}{3}\cdot\lambda o\gamma 0{,}08562=\overline{1}{,}64419$$

$$\lambda o\gamma\sigma\upsilon\nu 48^0 10'35''=\overline{1}{,}82402$$

$$\sigma\upsilon\lambda o\gamma\eta\mu 60^0 52'55''=0{,}05868$$

$$\sigma\upsilon\lambda o\gamma\varepsilon\varphi 36^0 43'26''=0{,}12725$$

Ἄρα : $\lambda o\gamma\sigma\varphi\theta =\overline{1}{,}65414=\lambda o\gamma\sigma\varphi(65^0 43'35'')$

Ὅθεν : $\theta=65^0 43'35''$ καὶ $x=k\cdot180^0-65^0 43'35''$

καὶ ἡ μικροτέρα θετικὴ τιμὴ τοῦ x ἀντιστοιχεῖ εἰς τὴν τιμὴν k=1. Δηλαδή :

$$x=114^0 16'25''.$$

153. Διὰ τῆς χρήσεως καταλλήλου βοηθητικῆς γωνίας, νὰ γίνουν λογισταὶ διὰ τῶν λογαρίθμων αἱ ἀκόλουθοι παραστάσεις :

1. $x=\sqrt{2}-1.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$x=\sqrt{2}-1=\sqrt{2}\left(1-\frac{\sqrt{2}}{2}\right)=\sqrt{2}(1-\sigma\upsilon\nu 45^0)=2\sqrt{2}\,\eta\mu^2 22^0{,}5.$$

2. $x=2+\sqrt{2}.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$x=2+\sqrt{2}=2\left(1+\frac{\sqrt{2}}{2}\right)=2(1+\sigma\upsilon\nu 45^0)=4\sigma\upsilon\nu^2 22^0{,}5.$$

3. $x=2+\sqrt{3}.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$x=2+\sqrt{3}=2\left(1+\frac{\sqrt{3}}{2}\right)=2(1+\sigma\upsilon\nu 30^0)=4\sigma\upsilon\nu^2 15^0.$$

4. $x=1-\sqrt{3}.$

Λύσις. Ἔχομεν διαδοχικῶς :

$$x=1-\sqrt{3}=2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\right)=2(\eta\mu 30^0-\eta\mu 60^0)=-4\eta\mu 15^0\sigma\upsilon\nu 45^0=-2\sqrt{2}\,\eta\mu 15^0$$

5. $$x = \sqrt{3} + \sqrt{2}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = \sqrt{3} + \sqrt{2} = 2\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\right) = 2(\eta\mu 60^\circ + \eta\mu 45^\circ) = 4\eta\mu 52^\circ,5\sigma\upsilon\nu 7^\circ,5.$$

6. $$x = 3 - \sqrt{3}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = 3 - \sqrt{3} = 3\left(1 - \frac{\sqrt{3}}{3}\right) = 3(1 - \varepsilon\varphi 30^\circ) = \frac{3\eta\mu 15^\circ}{\sigma\upsilon\nu 45^\circ \sigma\upsilon\nu 30^\circ} = 2\sqrt{6}\,\eta\mu 15^\circ$$

7. $$x = \frac{2+\sqrt{2}}{2-\sqrt{2}}$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = \frac{2+\sqrt{2}}{2-\sqrt{2}} = \frac{1+\frac{\sqrt{2}}{2}}{1-\frac{\sqrt{2}}{2}} = \frac{1+\sigma\upsilon\nu 45^\circ}{1-\sigma\upsilon\nu 45^\circ} = \sigma\varphi^2 22^\circ,5.$$

8. $$x = \frac{3-\sqrt{3}}{3+\sqrt{3}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = \frac{3-\sqrt{3}}{3+\sqrt{3}} = \frac{1-\frac{\sqrt{3}}{3}}{1+\frac{\sqrt{3}}{3}} = \frac{1-\varepsilon\varphi 30^\circ}{1+\varepsilon\varphi 30^\circ} = \varepsilon\varphi(45^\circ - 30^\circ) = \varepsilon\varphi 15^\circ.$$

9. $$x = \frac{\sqrt{3}+1}{\sqrt{3}-1}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{\varepsilon\varphi 60^\circ + 1}{\varepsilon\varphi 60^\circ - 1} = -\varepsilon\varphi(60^\circ + 45^\circ) = -\varepsilon\varphi(105^\circ) = \varepsilon\varphi 75^\circ.$$

154. Νὰ γίνουν λογισταὶ διὰ τῶν λογαρίθμων αἱ παραστάσεις:

1. $$x = 1 + 2\eta\mu\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = 1 + 2\eta\mu\alpha = 2\left(\frac{1}{2} + \eta\mu\alpha\right) = 2(\eta\mu 30^\circ + \eta\mu\alpha) = 4\eta\mu\left(15^\circ + \frac{\alpha}{2}\right)\sigma\upsilon\nu\left(15^\circ - \frac{\alpha}{2}\right)$$

2. $$x = 1 - 2\sigma\upsilon\nu\alpha.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x = 1 - 2\sigma\upsilon\nu\alpha = 2\left(\frac{1}{2} - \sigma\upsilon\nu\alpha\right) = 2(\sigma\upsilon\nu 60^\circ - \sigma\upsilon\nu\alpha) = 4\eta\mu\left(\frac{\alpha}{2} + 30^\circ\right)\eta\mu\left(\frac{\alpha}{2} - 30^\circ\right)$$

3. $\mathbf{x=1+\sqrt{2}\,\eta\mu\alpha}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=1+\sqrt{2}\,\eta\mu\alpha=\sqrt{2}\cdot\left(\frac{\sqrt{2}}{2}+\eta\mu\alpha\right)=\sqrt{2}(\eta\mu 45^0+\eta\mu\alpha)=$$

$$=2\sqrt{2}\,\eta\mu\left(22^0,5+\frac{\alpha}{2}\right)\sigma\upsilon\nu\left(22^0,5-\frac{\alpha}{2}\right).$$

4. $\mathbf{x=2\sigma\upsilon\nu\alpha-\sqrt{3}}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=2\sigma\upsilon\nu\alpha-\sqrt{3}=2\left(\sigma\upsilon\nu\alpha-\frac{\sqrt{3}}{2}\right)=2(\sigma\upsilon\nu\alpha-\sigma\upsilon\nu 30^0)=$$

$$=4\eta\mu\left(15^0+\frac{\alpha}{2}\right)\eta\mu\left(15^0-\frac{\alpha}{2}\right).$$

5. $\mathbf{x=1-\sqrt{3}\,\sigma\varphi\alpha}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=1-\sqrt{3}\,\sigma\varphi\alpha=\sqrt{3}\left(\frac{\sqrt{3}}{3}-\sigma\varphi\alpha\right)=\sqrt{3}\,(\sigma\varphi 60^0-\sigma\varphi\alpha)=$$

$$=\frac{\sqrt{3}\,\eta\mu(\alpha-60^0)}{\eta\mu 60^0\eta\mu\alpha}=\frac{2\eta\mu(\alpha-60^0)}{\eta\mu\alpha}.$$

6. $\mathbf{x=\eta\mu\alpha+\eta\mu 2\alpha+\eta\mu 3\alpha}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=(\eta\mu 3\alpha+\eta\mu\alpha)+\eta\mu 2\alpha=2\eta\mu 2\alpha\sigma\upsilon\nu\alpha+\eta\mu 2\alpha=2\eta\mu 2\alpha\left(\sigma\upsilon\nu\alpha+\frac{1}{2}\right)=$$

$$=2\eta\mu 2\alpha(\sigma\upsilon\nu\alpha+\sigma\upsilon\nu 60^0)=4\eta\mu 2\alpha\ \sigma\upsilon\nu\left(30^0+\frac{\alpha}{2}\right)\sigma\upsilon\nu\left(30^0-\frac{\alpha}{2}\right).$$

7. $\mathbf{x=2\sigma\upsilon\nu\alpha+\sqrt{3}\,\eta\mu\alpha}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=\sigma\upsilon\nu\alpha+\sqrt{3}\,\eta\mu\alpha=\sigma\upsilon\nu\alpha+\varepsilon\varphi 60^0\eta\mu\alpha=\sigma\upsilon\nu\alpha+\frac{\eta\mu 60^0}{\sigma\upsilon\nu 60^0}\eta\mu\alpha=$$

$$=(\sigma\upsilon\nu 60^0\sigma\upsilon\nu\alpha+\eta\mu 60^0\eta\mu\alpha)\frac{1}{\sigma\upsilon\nu 60^0}=\frac{\sigma\upsilon\nu(60^0-\alpha)}{\sigma\upsilon\nu 60^0}=2\sigma\upsilon\nu(60^0-\alpha).$$

8. $\mathbf{x=\dfrac{\sqrt{3}+\varepsilon\varphi\alpha}{1-\sqrt{3}\cdot\varepsilon\varphi\alpha}}$.

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=\frac{\sqrt{3}+\varepsilon\varphi\alpha}{1-\sqrt{3}\,\varepsilon\varphi\alpha}=\frac{\varepsilon\varphi 60^0+\varepsilon\varphi\alpha}{1-\varepsilon\varphi 60^0\varepsilon\varphi\alpha}=\varepsilon\varphi(60^0+\alpha).$$

155. Ἐὰν εἶναι γνωστοὶ οἱ λογα καὶ λογβ μὲ λογα>λογβ, νὰ γίνουν λογισταὶ διὰ τῶν λογαρίθμων αἱ παραστάσεις:

1. $$x=\sqrt{\alpha^2-\beta^2}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\frac{\beta}{\alpha}=\sigma\upsilon\nu\varphi$

$$x=\sqrt{\alpha^2-\beta^2}=\alpha\sqrt{1-\frac{\beta^2}{\alpha^2}}=\alpha\sqrt{1-\sigma\upsilon\nu^2\varphi}=\alpha\eta\mu\varphi.$$

2. $$x=\sqrt{\alpha+\beta}+\sqrt{\alpha-\beta}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\frac{\beta}{\alpha}=\sigma\upsilon\nu\varphi$

$$x=\sqrt{\alpha+\beta}+\sqrt{\alpha-\beta}=\sqrt{\alpha+\beta}\left(1+\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right)=\alpha\sqrt{1+\frac{\beta}{\alpha}}\left(1+\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}\right)=$$

$$=\alpha\sqrt{1+\frac{\beta}{\alpha}}\left(1+\sqrt{\frac{1-\frac{\beta}{\alpha}}{1+\frac{\beta}{\alpha}}}\right)=\alpha\sqrt{1+\sigma\upsilon\nu\varphi}\left(1+\sqrt{\frac{1-\sigma\upsilon\nu\varphi}{1+\sigma\upsilon\nu\varphi}}\right)=$$

$$=\alpha\sqrt{2}\,\sigma\upsilon\nu\frac{\varphi}{2}\left(1+\varepsilon\varphi\frac{\varphi}{2}\right)=\alpha\sqrt{2}\,\sigma\upsilon\nu\frac{\varphi}{2}\cdot\frac{\eta\mu\left(45^0+\frac{\varphi}{2}\right)}{\sigma\upsilon\nu45^0\sigma\upsilon\nu\frac{\varphi}{2}}=2\alpha\eta\mu\left(45^0+\frac{\varphi}{2}\right)$$

3. $$x=\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}+\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}.$$

Λύσις. Ἔχομεν διαδοχικῶς:

$$x=\sqrt{\frac{\alpha-\beta}{\alpha+\beta}}+\sqrt{\frac{\alpha+\beta}{\alpha-\beta}}=\sqrt{\frac{1-\frac{\beta}{\alpha}}{1+\frac{\beta}{\alpha}}}+\sqrt{\frac{1+\frac{\beta}{\alpha}}{1-\frac{\beta}{\alpha}}}=$$

$$=\sqrt{\frac{1-\sigma\upsilon\nu\varphi}{1+\sigma\upsilon\nu\varphi}}+\sqrt{\frac{1+\sigma\upsilon\nu\varphi}{1-\sigma\upsilon\nu\varphi}}=$$

$$=\varepsilon\varphi\frac{\varphi}{2}+\sigma\varphi\frac{\varphi}{2}=\frac{1}{\eta\mu\frac{\varphi}{2}\sigma\upsilon\nu\frac{\varphi}{2}}=\frac{2}{2\eta\mu\frac{\varphi}{2}\sigma\upsilon\nu\frac{\varphi}{2}}=\frac{2}{\eta\mu\varphi}.$$

4. $$x=\frac{4(\alpha-\beta)\sqrt{\alpha\beta}}{(\alpha+\beta)^2}.$$

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\frac{\beta}{\alpha}=\varepsilon\varphi^2\omega$

$$x=\frac{4(\alpha-\beta)\sqrt{\alpha\beta}}{(\alpha+\beta)^2}=\frac{4\left(1-\frac{\beta}{\alpha}\right)\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\right)^2}=\frac{4(1-\varepsilon\varphi^2\omega)\varepsilon\varphi\omega}{(1+\varepsilon\varphi^2\omega)^2}=$$

$$=\frac{4(1-\varepsilon\varphi^2\omega)\varepsilon\varphi\omega}{\left(\frac{1}{\sigma\upsilon\nu^2\omega}\right)^2}=4(\sigma\upsilon\nu^2\omega-\eta\mu^2\omega)\eta\mu\omega\sigma\upsilon\nu\omega=2\eta\mu2\omega\sigma\upsilon\nu2\omega=\eta\mu4\omega.$$

5. $$x=\sqrt{\alpha^2+\beta^2-\gamma^2}$$

Λύσις. Εἶναι: $x=\alpha\sqrt{\left(1+\frac{\beta^2}{\alpha^2}\right)-\frac{\gamma^2}{\alpha^2}}=\alpha\sqrt{\sigma\varphi\omega-\varepsilon\varphi\omega}=\alpha\sqrt{\frac{2\sigma\upsilon\nu2\omega}{\eta\mu2\omega}}$.

156. Ἐὰν $\left.\begin{array}{l}\alpha=108{,}7\\ \beta=73{,}45\end{array}\right\}$, νὰ ὑπολογισθῇ ἡ $x=\sqrt{\alpha^2+\beta^2}$.

Λύσις. Ἔχομεν διαδοχικῶς, ἂν $\frac{\beta}{\alpha}=\varepsilon\varphi\omega$,

$$x=\sqrt{\alpha^2+\beta^2}=\alpha\sqrt{1+\frac{\beta^2}{\alpha^2}}=\alpha\sqrt{1+\varepsilon\varphi^2\omega}=\frac{\alpha}{\sigma\upsilon\nu\omega}$$

καὶ $\lambda o\gamma\varepsilon\varphi\omega=\lambda o\gamma\frac{\beta}{\alpha}=\lambda o\gamma\beta-\lambda o\gamma\alpha=\lambda o\gamma 73{,}45-\lambda o\gamma 108{,}7=$

$=\lambda o\gamma 73{,}45+\sigma\upsilon\lambda o\gamma 108{,}7=1{,}86599+\bar{3},96377=\bar{1},82976$

ἐξ οὗ: $\omega=34^\circ 2' 51''$.

Ὅθεν: $\lambda o\gamma x=\lambda o\gamma\left(\frac{\alpha}{\sigma\upsilon\nu\omega}\right)=\lambda o\gamma\alpha+\sigma\upsilon\lambda o\gamma\sigma\upsilon\nu\omega=2{,}03623+0{,}08167=2{,}11790$

ἐξ οὗ: $x=131{,}19$,

157. Ἐὰν $\left.\begin{array}{l}\alpha=71{,}29\\ \beta=32{,}57\end{array}\right\}$, νὰ ὑπολογισθῇ ἡ: $x=\sqrt{\alpha^2-\beta^2}$.

Λύσις. Ἐὰν $\frac{\beta}{\alpha}=\sigma\upsilon\nu\omega$, θὰ ἔχωμεν:

$$x=\sqrt{\alpha^2-\beta^2}=\alpha\sqrt{1-\frac{\beta^2}{\alpha^2}}=\alpha\sqrt{1-\sigma\upsilon\nu^2\omega}=\alpha\eta\mu\omega$$

καὶ $\lambda o\gamma\sigma\upsilon\nu\omega=\lambda o\gamma\left(\frac{\beta}{\alpha}\right)=\lambda o\gamma\beta+\sigma\upsilon\lambda o\gamma\alpha=1{,}51282+\bar{2},14697=\bar{1},65979$

ἐξ οὗ: $\omega=62^\circ 48' 53''$

καὶ $\lambda o\gamma x=\lambda o\gamma\alpha+\lambda o\gamma\eta\mu\omega=1{,}85303+\bar{1},94916=1{,}80219$

ἐξ οὗ: $x=63{,}414$.

158. Ἐὰν $\alpha=4258$, $\beta=3672$ καὶ $\beta\varepsilon\varphi 3x=\alpha+\sqrt{\alpha^2+\beta^2}$, νὰ ὑπολογισθῇ ὁ x, ὥστε νὰ εἶναι: $0^\circ<x<180^\circ$.

Λύσις. Θέτομεν $\frac{\alpha}{\beta}=\varepsilon\varphi\varphi$ καὶ ἔχομεν:

$$\varepsilon\varphi 3x=\frac{\alpha}{\beta}+\sqrt{1+\frac{\alpha^2}{\beta^2}}=\varepsilon\varphi\varphi+\frac{1}{\sigma\upsilon\nu\varphi}=\frac{1+\eta\mu\varphi}{\sigma\upsilon\nu\varphi}=\varepsilon\varphi\left(45^\circ+\frac{\varphi}{2}\right).$$

Εἶναι δέ : $\lambda o \gamma \varepsilon \varphi \varphi = \left(\frac{\alpha}{\beta}\right) = \lambda o \gamma \alpha + \sigma \upsilon \lambda o \gamma \beta = 0{,}06431$, ἐξ οὗ $\varphi = 49^0 13' 36''$

ὅθεν καί : $$45^0 + \frac{\varphi}{2} = 69^0 36' 48''.$$

Ἄρα : $3x = k \cdot 180^0 + 69^0 36' 48''$ ἤ $x = k \cdot 60^0 + 23^0 12' 16''$

Διὰ $k = 0,1,2$, λαμβάνομεν ἀντιστοίχως :

$$\left.\begin{array}{l} x = 23^0 12' 16'' \\ x = 83^0 12' 16'' \\ x = 143^0 12' 16'' \end{array}\right\}$$

159. Ἐὰν $\alpha = 4625{,}5$, $\beta = 3944{,}6$, $\theta = 51^0 57' 44''$, $\theta_1 = 63^0 18' 27''$

καί : $$\varepsilon\varphi 2x = \frac{\alpha \eta\mu\theta_1 - \beta \eta\mu\theta}{\alpha \eta\mu\theta_1 + \beta \eta\mu\theta},$$

νὰ ὑπολογισθῇ ὁ x, ἵνα $0^0 < x < 180^0$.

Λύσις. Θέτομεν $\frac{\beta \eta\mu\theta}{\alpha \eta\mu\theta_1} = \varepsilon\varphi\omega$, ἄρα : $\varepsilon\varphi 2x = \frac{1 - \varepsilon\varphi\omega}{1 + \varepsilon\varphi\omega} = \varepsilon\varphi(45^0 - \omega)$

καὶ $$\lambda o \gamma \varepsilon\varphi\omega = \lambda o \gamma \beta + \lambda o \gamma \eta\mu\theta + \sigma\upsilon\lambda o \gamma \alpha + \sigma\upsilon\lambda o \gamma \eta\mu\theta_1 =$$
$$= 3{,}59601 + \overline{1}{,}89631 + \overline{4}{,}33465 + 0{,}04894 = \overline{1}{,}87591$$

ἐξ οὗ : $\omega = 36^0 55' 25''$. Ἄρα $45^0 - \omega = 8^0 4' 35''$ καὶ

$$2x = k \cdot 180^0 + 8^0 4' 35'' \quad \text{ἤ} \quad x = k \cdot 90^0 + 4^0 2' 18''.$$

Διὰ $k = 0,1$, ἔχομεν : $x = 4^0 2' 18''$ καὶ $x = 94^0 2' 18''$.

160. Νὰ ἐπιλυθῇ ἡ ἐξίσωσις : $8x^2 - 36{,}75x - 25{,}628 = 0$.

Λύσις. Ἡ δοθεῖσα ἐξίσωσις ἐπιδέχεται δύο ρίζας ἑτεροσήμους, καθόσον τὸ γινόμενον τοῦ συντελεστοῦ τοῦ x^2 καὶ τοῦ γνωστοῦ ὅρου εἶναι ἀρνητικόν.

Εἶναι τῆς μορφῆς $\alpha x^2 - \beta x - \gamma = 0$ καὶ ἔχει ρίζας, αἱ ὁποῖαι δίδονται ὑπὸ τῶν τύπων :

$$x_1 = -\sqrt{\frac{\gamma}{\alpha}}\, \varepsilon\varphi \frac{\varphi}{2} \quad \text{καὶ} \quad x_2 = \sqrt{\frac{\gamma}{\alpha}}\, \sigma\varphi \frac{\varphi}{2}$$

ἔνθα : $$\varepsilon\varphi^2\varphi = \frac{4\alpha\gamma}{\beta^2}.$$

Θὰ εἶναι : $$\lambda o \gamma \varepsilon\varphi\varphi = \frac{1}{2}[\lambda o \gamma 4 + \lambda o \gamma \alpha + \lambda o \gamma \gamma] + \sigma\upsilon\lambda o \gamma \beta =$$

$$= \frac{1}{2}[0{,}60206 + 0{,}90309 + 1{,}41049] + \overline{2}{,}43474 = \overline{1}{,}89256$$

ἐξ οὗ $\varphi = 37^0 59'$ καὶ $\frac{\varphi}{2} = 18^0 59' 30''$. Κατ' ἀκολουθίαν :

$$\lambda o\gamma(-x_1)=\lambda o\gamma\left[\sqrt{\frac{\gamma}{\alpha}}\ \varepsilon\varphi\ \frac{\varphi}{2}\right]=\frac{1}{2}\ [\lambda o\gamma\gamma-\lambda o\gamma\alpha]+\lambda o\gamma\varepsilon\varphi\ \frac{\varphi}{2}$$

$$=\frac{1}{2}\ (1{,}41049-0{,}90309)+\lambda o\gamma\varepsilon\varphi 18^{0}59'30''$$

$$=0{,}25370+\overline{1},53677=\overline{1},79047,$$

ἐξ οὗ: $-x_1=0{,}617257$ ἤ $x_1=-0{,}617257$

καί: $\lambda o\gamma x_2=\lambda o\gamma\left[\sqrt{\frac{\gamma}{\alpha}}\ \sigma\varphi\ \frac{\varphi}{2}\right]=\frac{1}{2}\ (\lambda o\gamma\gamma-\lambda o\gamma\alpha)+\lambda o\gamma\sigma\varphi\ \frac{\varphi}{2}=$

$$=\frac{1}{2}\ (\lambda o\gamma\gamma-\lambda o\gamma\alpha)+\lambda o\gamma\sigma\varphi 18^{0}59'30''=0{,}25370+0{,}46323=0{,}71693$$

ἐξ οὗ: $x=5{,}211.$

Ὥστε: $x_1=-0{,}617247$ καὶ $x_2=5{,}311.$

161. Νὰ ἐπιλυθοῦν αἱ ἐξισώσεις:

1. $x^2-148{,}7x+1385=0$
2. $x^2-245{,}7x-1217{,}6=0$
3. $x^2+16{,}73x-64{,}53=0$
4. $x^2+75{,}23x-433{,}7=0$-

Λύσις. Ἐργαζόμεθα ὅπως καὶ εἰς τὴν προηγουμένην ἄσκησιν.

162. Ἐὰν $2\eta\mu x=\eta\mu\alpha+\eta\mu(\alpha+\omega)+\eta\mu(\alpha+2\omega)$ καὶ $\alpha=18^{0}25'37''$, $\omega=7^{0}17'26''$, νὰ ὑπολογισθῇ ὁ x.

Λύσις. Ἔχομεν:

$$2\eta\mu x=[\eta\mu(\alpha+2\omega)+\eta\mu\alpha]+\eta\mu(\alpha+\omega)$$

$$=2\eta\mu(\alpha+\omega)\sigma\upsilon\nu\omega+\eta\mu(\alpha+\omega)$$

$$=2\eta\mu(\alpha+\omega)\ \left(\sigma\upsilon\nu\omega+\frac{1}{2}\right)=2\eta\mu(\alpha+\omega)(\sigma\upsilon\nu\omega+\sigma\upsilon\nu 60^{0})=$$

$$=4\eta\mu(\alpha+\omega)\sigma\upsilon\nu\ \left(30^{0}+\frac{\omega}{2}\right)\sigma\upsilon\nu\ \left(30^{0}-\frac{\omega}{2}\right)$$

ἐξ οὗ: $\eta\mu x=2\eta\mu(\alpha+\omega)\sigma\upsilon\nu\ \left(30^{0}+\frac{\omega}{2}\right)\sigma\upsilon\nu\ \left(30^{0}-\frac{\omega}{2}\right)=$

$$=2\eta\mu(25^{0}43'3'')\sigma\upsilon\nu(33^{0}38'43'')\sigma\upsilon\nu(26^{0}21'17'').$$

Ἄρα: $\lambda o\gamma\eta\mu x=\lambda o\gamma 2+\lambda o\gamma\eta\mu(25^{0}43'3'')+\lambda o\gamma\sigma\upsilon\nu(33^{0}38'43'')+$
$+\lambda o\gamma\sigma\upsilon\nu(26^{0}21'17'')=$

$$=0{,}30103+\overline{1},63742+\overline{1},92038+\overline{1},95234=\overline{1},81117$$

ἐξ οὗ: $x=40^{0}20'52''.$

163. Νὰ ὑπολογισθῇ ὁ x, οὕτως ὥστε: $x^3=\alpha^3\eta\mu\theta+\beta^3\sigma\upsilon\nu\theta$, ἂν $\alpha=18928$, $\beta=20842$ καὶ $\theta=115^{0}45'27''$.

Λύσις. Θέτομεν $\varepsilon\varphi\omega=\frac{\beta^3}{\alpha^3}$ καὶ ἔχομεν διαδοχικῶς:

$$x^3=\alpha^3\left(\eta\mu\theta+\frac{\beta^3}{\alpha^3}\ \sigma\upsilon\nu\theta\right)=\alpha^3(\eta\mu\theta+\varepsilon\varphi\omega\cdot\sigma\upsilon\nu\theta)=\alpha^3\left(\eta\mu\theta+\frac{\eta\mu\omega}{\sigma\upsilon\nu\omega}\ \sigma\upsilon\nu\theta\right)=$$

$$=\frac{\alpha^3}{\sigma\upsilon\nu\omega}\ (\sigma\upsilon\nu\omega\eta\mu\theta+\eta\mu\omega\sigma\upsilon\nu\theta)=\frac{\alpha^3\eta\mu(\omega+\theta)}{\sigma\upsilon\nu\omega}. \qquad (1)$$

'Αλλά: $\lambda o\gamma\varepsilon\varphi\omega=\lambda o\gamma\left(\frac{\beta^3}{\alpha^3}\right)=3\lambda o\gamma\beta-3\lambda o\gamma\alpha=3(\lambda o\gamma\beta-\lambda o\gamma\alpha)=$

$$=3(4{,}31894-4{,}27710)=0{,}12552,$$

ἐξ οὗ: $\omega=53^0 10'$ ὁπότε ἡ (1) γίνεται:

$$x^3=\frac{\alpha^3\eta\mu(53^0 10'+115^0 45'27'')}{\sigma\upsilon\nu(53^0 10')}=\frac{\alpha^3\eta\mu(168^0 55'27'')}{\sigma\upsilon\nu(53^0 10')}=\frac{\alpha^3\eta\mu(11^0 4'43'')}{\sigma\upsilon\nu(53^0 10')}.$$

Ἄρα: $3\lambda o\gamma x=3\lambda o\gamma\alpha+\lambda o\gamma\eta\mu(11^0 4'43'')-\lambda o\gamma\sigma\upsilon\nu(53^0 10')$

$$=3\cdot 4{,}27710+\overline{1}{,}28366-\overline{1}{,}77778$$

$$=12{,}83130+\overline{1}{,}28366+0{,}22222=12{,}33718$$

ἢ $\lambda o\gamma x=12{,}33718$, ἐξ οὗ: $x=2173600000000.$

164. Νὰ ὑπολογισθοῦν αἱ μεταξὺ 0^0 καὶ 180^0 τιμαὶ τοῦ x, αἵτινες ἐπαληθεύουν τὴν ἐξίσωσιν: $\varepsilon\varphi 3x=\dfrac{\alpha+\sqrt{\alpha^2+\beta^2}}{\beta}$ (1)

ἂν $\alpha=4167$ καὶ $\beta=3582{,}4$.

Λύσις. Ἡ δοθεῖσα ἐξίσωσις γράφεται:

$$\varepsilon\varphi 3x=\frac{\alpha}{\beta}\left[1+\sqrt{1+\left(\frac{\beta}{\alpha}\right)^2}\ \right] \qquad (2)$$

Θέτομεν $\frac{\beta}{\alpha}=\varepsilon\varphi\varphi$ καὶ ἔχομεν:

$$\varepsilon\varphi 3x=\sigma\varphi\varphi[1+\sqrt{1+\varepsilon\varphi^2\varphi}\,]=\sigma\varphi\varphi\left(1+\frac{1}{\sigma\upsilon\nu\varphi}\right)=\frac{\sigma\upsilon\nu\varphi+1}{\eta\mu\varphi}=$$

$$=\frac{2\sigma\upsilon\nu^2\frac{\varphi}{2}}{2\eta\mu\frac{\varphi}{2}\sigma\upsilon\nu\frac{\varphi}{2}}=\sigma\varphi\frac{\varphi}{2}. \qquad (3)$$

'Αλλὰ $\lambda o\gamma\varepsilon\varphi\varphi=\lambda o\gamma\left(\frac{\beta}{\alpha}\right)=\lambda o\gamma\beta-\lambda o\gamma\alpha=3{,}55418-3{,}61982=\overline{1}{,}93436$

ἐξ οὗ: $\varphi=40^0 41'11''$ καὶ $\frac{\varphi}{2}=20^0 20'35'',5.$

Κατ' ἀκολουθίαν: $\varepsilon\varphi 3x=\sigma\varphi\frac{\varphi}{2}=\varepsilon\varphi\left(90^0-\frac{\varphi}{2}\right)$

ἢ $3x=k\cdot 180^0+90^0-\frac{\varphi}{2} \Longrightarrow x=k\cdot 60^0+30^0-\frac{\varphi}{6}$ (4)

'Αλλὰ $\frac{\varphi}{6}=6^0 46'54'',8$ καὶ $30^0-\frac{\varphi}{6}=(29^0 59'60''-6^0 46'54'',8)=23^0 13'8'',2.$

Διὰ k=0, 1, 2, ἐκ τῆς (4) λαμβάνομεν ἀντιστοίχως:

$x=23^0 13'8'',2$ $x=83^0 13'8'',2$ $x=143^0 13'8'',2.$

ΠΑΡΑΡΤΗΜΑ

ΠΡΟΒΛΗΜΑ Ι.—Νὰ γίνῃ γινόμενον παραγόντων ἡ παράστασις:

$$x^{2\nu}-2x^{\nu}\,\text{συν}(\nu\theta)+1.$$

Ἐπιλύομεν τὴν ἐξίσωσιν:

$$x^{2\nu}-2x^{\nu}\,\text{συν}(\nu\theta)+1=0 \quad \text{ἤ} \quad x^{2\nu}-2x^{\nu}\,\text{συν}(\nu\theta)+\text{συν}^2(\nu\theta)=-\text{ημ}^2(\nu\theta)$$

$$\text{ἤ} \quad [x^{\nu}-\text{συν}(\nu\theta)]^2=-\text{ημ}^2(\nu\theta), \quad \text{ἐξ οὗ:} \quad x^{\nu}-\text{συν}(\nu\theta)=\pm i\,\text{ημ}(\nu\theta)$$

$$\text{ἤ} \quad x^{\nu}=\text{συν}(\nu\theta)\pm i\,\text{ημ}(\nu\theta) \Longrightarrow x=[\text{συν}(\nu\theta)\pm i\,\text{ημ}(\nu\theta)]^{\frac{1}{\nu}}. \qquad (1)$$

Αἱ ρίζαι τῆς (2), κατὰ τὰ γνωστὰ ἐκ τῆς Ἀλγέβρας, εἶναι:

$$\text{συν}\theta\pm i\,\text{ημ}\theta,\ \text{συν}\left(\theta+\frac{2\pi}{\nu}\right)\pm i\,\text{ημ}\left(\theta+\frac{2\pi}{\nu}\right),\ \text{συν}\left(\theta+\frac{4\pi}{\nu}\right)\pm i\,\text{ημ}\left(\theta+\frac{4\pi}{\nu}\right),$$

$$\ldots,\ \text{συν}\left[\theta+\frac{2(\nu-1)\pi}{\nu}\right]\pm i\,\text{ημ}\left[\theta+\frac{2(\nu-1)\pi}{\nu}\right],$$

καὶ εἶναι 2ν κατὰ τὸ πλῆθος.

Κατὰ τὰ γνωστὰ ἐκ τῆς Ἀλγέβρας θὰ εἶναι:

$$x^{2\nu}-2x^{\nu}\,\text{συν}(\nu\theta)+1=(x-\text{συν}\theta-i\,\text{ημ}\theta)(x-\text{συν}\theta+i\,\text{ημ}\theta)$$

$$\left[x-\text{συν}\left(\theta+\frac{2\pi}{\nu}\right)-i\,\text{ημ}\left(\theta+\frac{2\pi}{\nu}\right)\right]\cdot\left[x-\text{συν}\left(\theta+\frac{2\pi}{\nu}\right)+i\,\text{ημ}\left(\theta+\frac{2\pi}{\nu}\right)\right]\cdot$$

$$\ldots\left\{x-\text{συν}\left[\theta+\frac{2(\nu-1)\pi}{\nu}\right]\pm i\,\text{ημ}\left[\theta+\frac{2(\nu-1)\pi}{\nu}\right]\right\}=$$

$$=(x^2-2x\,\text{συν}\theta+1)\left[x^2-2x\,\text{συν}\left(\theta+\frac{2\pi}{\nu}\right)+1\right]\left[x^2-2x\,\text{συν}\left(\theta+\frac{4\pi}{\nu}\right)+1\right]$$

$$\ldots\left[x^2-2x\,\text{συν}\left(\theta+\frac{2\nu-2}{\nu}\pi\right)+1\right]. \qquad (3)$$

Διαιροῦντες ἀμφότερα τὰ μέλη τῆς (3) διὰ x^{ν}, λαμβάνομεν:

$$x^{\nu}+\frac{1}{x^{\nu}}-2\text{συν}(\nu\theta)=\left(x+\frac{1}{x}-2\text{συν}\theta\right)\left[x+\frac{1}{x}-2\text{συν}\left(\theta+\frac{2\pi}{\nu}\right)\right]$$

$$\ldots\left\{x+\frac{1}{x}-2\text{συν}\left(\theta+\frac{2\nu-2}{\nu}\pi\right)\right\} \qquad (4)$$

Ἡ (4) γράφεται καὶ ὡς ἑξῆς:

$$x^{\nu}+\frac{1}{x^{\nu}}-2\text{συν}(\nu\theta)=\prod_{\lambda=0}^{\lambda=\nu-1}\left[x+\frac{1}{x}-2\text{συν}\left(\theta+\frac{2\lambda\pi}{\nu}\right)\right] \qquad (5)$$

ΠΡΟΒΛΗΜΑ ΙΙ.—Νὰ ἐπιλυθῇ ἡ ἐξίσωσις: $x^{\nu}-1=0.$ (1)

Λύσις. Εἶναι: $x^{\nu}=1=\text{συν}(2\lambda\pi)\pm i\,\text{ημ}(2\lambda\pi),\ \lambda\in\mathbf{Z}.$

α) Ἂν ν=ἄρτιος, τότε ἐργαζόμενοι ὡς ἀνωτέρω εὑρίσκομεν:

$$x^\nu - 1 = (x^2-1)\left(x^2-2x\sigma\upsilon\nu\frac{2\pi}{\nu}+1\right)\left(x^2-2x\sigma\upsilon\nu\frac{4\pi}{\nu}+1\right)\cdots$$

$$\cdots\left(x^2-2x\sigma\upsilon\nu\frac{\nu-2}{\nu}\pi+1\right). \tag{2}$$

β) Ἂν ν=περιττός, τότε θὰ εἶναι:

$$x^\nu - 1 = (x-1)\left(x^2-2x\sigma\upsilon\nu\frac{2\pi}{\nu}+1\right)\left(x^2-2x\sigma\upsilon\nu\frac{4\pi}{\nu}+1\right)\cdots\cdot$$

$$\cdots\left(x^2-2x\sigma\upsilon\nu\frac{\nu-1}{\nu}\right). \tag{3}$$

Ἐκ τῆς (3) ἕπεται ὅτι:

$$\frac{x^\nu-1}{x-1} = \left(x^2-2x\sigma\upsilon\nu\frac{2\pi}{\nu}+1\right)\left(x^2-2x\sigma\upsilon\nu\frac{4\pi}{\nu}+1\right)\cdots\left(x^2-2x\sigma\upsilon\nu\frac{\nu-1}{\nu}\pi+1\right)$$

ἐκ $\frac{\nu-1}{2}$ παραγόντων.

ἢ $$x^{\nu-1}+x^{\nu-2}+\ldots+1 = \left(x^2-2x\sigma\upsilon\nu\frac{2\pi}{\nu}+1\right)\left(x^2-2x\sigma\upsilon\nu\frac{4\pi}{\nu}+1\right)\cdots$$

$$\cdots\left(x^2-2x\sigma\upsilon\nu\frac{\nu-1}{\nu}\pi+1\right). \tag{4}$$

Διὰ $x=1$, ἡ (4) γίνεται:

$$\nu = \left(2-2\sigma\upsilon\nu\frac{2\pi}{\nu}\right)\left(2-2\sigma\upsilon\nu\frac{4\pi}{\nu}\right)\cdots\left(2-2\sigma\upsilon\nu\frac{\nu-1}{\nu}\pi\right)$$

$$= 4\eta\mu^2\frac{2\pi}{2\nu}\cdot 4\eta\mu^2\frac{4\pi}{2\nu}\cdots\cdot 4\eta\mu^2\frac{\nu-1}{2\nu}\pi$$

ἐξ οὗ: $$\sqrt{\nu} = 2^{\frac{\nu-1}{2}}\eta\mu\frac{2\pi}{2\nu}\eta\mu\frac{4\pi}{2\nu}\cdots\eta\mu\frac{\nu-1}{2\nu}\pi$$

ἢ $$\sqrt{\nu} = 2^{\frac{\nu-1}{\nu}}\eta\mu\frac{\pi}{\nu}\eta\mu\frac{2\pi}{\nu}\cdots\eta\mu\frac{\nu-1}{2\nu}\pi \tag{5}$$

Ἐὰν εἰς τὴν (5) τεθῇ ἀντὶ τοῦ ν τὸ $2\nu+1$, θὰ ἔχωμεν:

$$\sqrt{2\nu+1} = 2^\nu\eta\mu\frac{\pi}{2\nu+1}\eta\mu\frac{2\pi}{2\nu+1}\cdots\eta\mu\frac{\nu\pi}{2\nu+1}$$

ἐξ οὗ: $$\eta\mu\frac{\pi}{2\nu+1}\eta\mu\frac{2\pi}{2\nu+1}\eta\mu\frac{3\pi}{2\nu+1}\cdots\eta\mu\frac{\nu\pi}{2\nu+1} = \frac{\sqrt{2\nu+1}}{2^\nu} \tag{6}$$

Κατ᾿ ἀνάλογον τρόπον ἐργαζόμενοι, εὑρίσκομεν ἐκ τῆς ἐξισώσεως $x^\nu+1=0$, ὅτι:

$$\frac{x^\nu+1}{x+1} = \left(x^2-2x\sigma\upsilon\nu\frac{\pi}{\nu}+1\right)\left(x^2-2x\sigma\upsilon\nu\frac{3\pi}{\nu}+1\right)$$

$$\cdots\left[x^2-2x\sigma\upsilon\nu\frac{(\nu-2)\pi}{\nu}+1\right]. \tag{7}$$

καὶ διὰ x=1, λαμβάνομεν:

$$1=\left(2-2συν\frac{π}{ν}\right)\left(2-2συν\frac{3π}{ν}\right)\cdots\left[2-2συν\frac{(ν-2)π}{ν}\right]$$

$$=4ημ^2\frac{π}{2ν}\cdot 4ημ^2\frac{3π}{2ν}\cdots 4ημ^2\frac{(ν-2)π}{2ν},\quad \text{ἐκ } \frac{ν-1}{2} \text{ παραγόντων}$$

ἢ

$$1=2^{\frac{ν-1}{2}}ημ\frac{π}{2ν}ημ\frac{3π}{2ν}\cdots ημ\frac{ν-2}{2ν}π$$

$$=2^{\frac{ν-1}{2}}συν\left(\frac{π}{2}-\frac{π}{2ν}\right)συν\left(\frac{π}{2}-\frac{3π}{2ν}\right)\cdots συν\left(\frac{π}{2}-\frac{ν-2}{2ν}π\right)$$

$$=2^{\frac{ν-1}{2}}συν\frac{π}{ν}συν\frac{2π}{ν}\cdots συν\frac{ν-1}{2ν}π \qquad (8)$$

Ἐὰν τεθῇ ἀντὶ τοῦ ν τὸ 2ν+1, ἡ (8) γίνεται:

$$1=2^{ν}συν\frac{π}{2ν+1}συν\frac{2π}{2ν+1}\cdots συν\frac{νπ}{2ν+1}$$

ἐξ οὗ:

$$συν\frac{π}{2ν+1}συν\frac{2π}{2ν+1}συν\frac{3π}{2ν+1}\cdots συν\frac{νπ}{2ν+1}=\frac{1}{2^{ν}} \qquad (9)$$

Ἐκ τῶν (6) καὶ (9) ἕπεται ὅτι:

$$εφ\frac{π}{2ν+1}εφ\frac{2π}{2ν+1}εφ\frac{3π}{2ν+1}\cdots εφ\frac{νπ}{2ν+1}=\sqrt{2ν+1} \qquad (10)$$

Αἱ (6), (9), (10) ἀποτελοῦν τὰς λύσεις τῶν ὑπ' ἀριθ. 4—5—6 τῆς ἀσκήσεως 85.

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